# Set, Relation and Functions

## Set

Set :- The collection of well defined object (elements) which are connected by some rules i.e. the given object $\in$ set.

eg - (i) A bunch of flowers
(ii) A line is set of points
(iii) A plane is set of lines
(iv) A pack of cards
(v) A bunch of graphes
(vi) The set of natural no.
(vii) The set of integers no.



Representation of set :-

(i) Tabular / Roster form ⇒

eg- $N = \{1, 2, 3 \ldots\}$
$W = \{0, 1, 2, 3 \ldots\}$
$Z = \{\ldots -2, -1, 0, 1, 2 \ldots\}$
$Z^+ = \{1, 2, 3 \ldots\}$
$Z^- = \{-1, -2, -3 \ldots\}$
$Q = \{1, 2, 3/2 \ldots\}$

(ii) Set Builder / Property form ⇒

$S = \{x \mid x \in p\}$

Eg -(i) The set of all letter in the word
STATISTICS $= \{A, S, T, I, C\}$

(ii) The set $A = \{1, 2, 3, 4, 5, 6, 7, 8,\}$ then
$A = \{ x \mid x \in N$ and $x < 9 \}$

Type of Sets ⇒

(i) Empty set / void set :- The set which has no element.
It is denoted by $\phi$ or $\{\}$.

(ii) Equal set :- If the two set are having same element.

eg- $S = \{a, e, i\}$ , $T = \{i, e, a\}$
$a \in S$ & $a \in T$
$e \in S$ & $e \in T$ so $\boxed{S = T}$
$i \in S$ & $i \in T$

(iii) Unequal set :- $\boxed{S \neq T}$
eg - $S = \{a, e, i\}$ , $T = \{a, b, e\}$
$a \in S$ & $a \in T$
$e \in S$ & $e \in T$ then $\boxed{S \neq T}$
$i \in S$ & $i \notin T$
$b \notin S$ & $b \in T$

(IV) Subset :- Every element of T is an element of S then we say T is subset of S and it is represented by
$T \subseteq S$ if $x \in T$ & $x \in S$

(v) Proper subset $\Rightarrow$ If $T \subset S$ but $T \neq S$

eg - $S = \{1, 2, 3\}$ , $T = \{1, 2\}$

where $3 \in S$ but $3 \notin T$ therefore

$T \subset S$ but $T \neq S$

(vi) Super subset $\rightarrow$ If $\boxed{S \supset T}$

Properties:—

(i) Every set is the subset of itself.

$S \subseteq S$

(ii) The null set is the subset of everyset.

$\emptyset \subseteq S$

(iii) If S is the subset of T & $T \subseteq K$ then

$S \subseteq K$

(iv) If finite set having n element then total subset $2^n$.

Eg-Q. Find subset for A set $A = \{1, 2, 3\}$

$n = 3$

$2^3 = 8$

$S = \{ \emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\} \}$

Q. $A = \{ \emptyset, 1, \{3\} \}$

$n = 3$

$2^3 = 8$

$A = \{ \emptyset, \{\emptyset\}, \{1\}, \{\{3\}\}, \{\emptyset, 1\}, \{\emptyset, \{3\}\}, \{1, \{3\}\},$
$\{\emptyset, 1, \{3\}\} \}$

Q. If $S = \{0, 1, 2, 3\}$ & $T = \{-1, 1\}$ then two set S&T subset or not?

$T \not\subseteq S$ & $S \not\subseteq T$

Q. $S = \{0, 1, 2, 3\}$

$n = 4$ then $2^4 = 16$

$B = \{\phi, \{0\}, \{1\}, \{2\}, \{3\}, \{0,1\}, \{0,2\}, \{0,3\}, \{1,2\}, \{1,3\},$
$\{2,3\}, \{0,1,2\}, \{0,1,3\}, \{0,2,3\}, \{1,2,3\}$
$\{0,1,2,3\} \}$

Null set :- A set having no elements. Denoted by $\phi$ or $\{\}$.

eg - $S = \{x : x \in R$ and $x^2 + 2 = 0\} = \phi$

this is an empty set as there is no any real no. which satisfied the given condition

eg :- $S = \{x \mid x \in R$ and $x > 3$ and also $x < 2\} = \phi$

Finite set :-

$S = \{x : x \in Z^+ \;\&\; x \le 9\}$

$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

Infinite set :-

$S = \{x \mid x \in Z \;\&\; x > 9\}$

i.e. $S = \{10, 11, 12 ---- \infty\}$

Cardinality of a finite set :-

The no. of element in the finite set S is called cardinal no. & it is denoted by $n(S)$ or $|S|$.

Set of sets :- A set itself may sometime be an element of another set then the later set is called the set of sets.

eg - The set of all lines in a plane since the line itself is nothing but a set of points.

Singleton set or Singlet :—
A set having only one element.
eg :- $\{6\}$, $\{\phi\}$
eg- A set of integer b/w 6 & 8 is $\{7\}$

Universal set :- If all the sets under consideration are subset of a fixed ~~conset~~ set then this fixed set is called universal set & denoted by U.
eg- If $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9 ...\}$
$A = \{1, 3, 5, 7 ...\}$ $B = \{2, 4, 6, 8 --\}$

Complement of a set :- If U be the universal set then complement of set S denoted by $S'$ or $S^c$ & this is denoted as $U - S$.

$U - S = S' = S^c = \{x : x \in U \text{ but } x \notin S\}$

Properties of complement of a set :—

(i) $U' = \phi$ (3.) $A \cup A' = U$
(ii) $\phi' = U$ (4.) $(A')' = A$
(5.) $A \cap A' = \phi$ ~~(6.) AUP~~

(6.) $(A \cup B)' = A' \cap B'$
(7.) $(A \cap B)' = A' \cup B'$
] De morgone property

Q. Prove that $(A \cup B)' = A' \cap B'$
Sol → To prove $(A \cup B)' \subseteq A' \cap B'$
$A' \cap B' \subseteq (A \cup B)'$

Let $x \in (A \cup B)'$ from $q^n$ — ①

$x \notin (A \cup B)$

$x \notin A$ and $x \notin B$

$x \notin A$ and $x \in B'$

$x \in A'$ and $x \in B'$

$x \in A' \cap B'$

Again :— $A' \cap B' \subset (A \cup B)'$

let $x \in A' \cap B'$

$x \notin A$ and $x \notin B$

$x \notin A \cup B$

$x \in (A \cup B)'$



Difference of 2 sets i.e. $(S-T)$

$S-T = \{ x : x \in S$ but $x \notin T \}$

Similarly

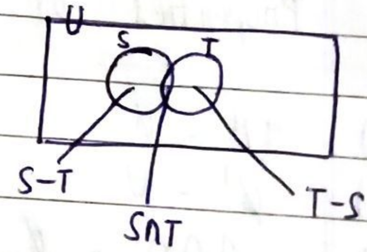
$T-S = \{ x : x \in T$ but $x \notin S \}$

eg :- If $S = \{1, 2, 3, 4, 5, 6\}$

$T = \{5, 6, 7, 8, 9, 10\}$

$S-T = \{1, 2, 3, 4\}$

$T-S = \{7, 8, 9, 10\}$




Imp Symmetric difference of two sets :—

⊕ or $\Delta$

$S \oplus T = (S-T) \cup (T-S) = (S \cup T) - (S \cap T)$ .

$S \oplus T = \{ x : x \in S$ and $x \notin T$ or $x \in T$ but $x \notin S \}$

eg - If $S = \{1, 2, 3, 4\}$ and $T = \{4, 5, 6, 7\}$ then

$S \oplus T = \{1, 2, 3, 5, 6, 7\}$

Power set :- The set of all subset of a given set S is called the power set of S and it is denoted by P(S).

$P(S) = \{T : T \subseteq S\}$ & $\phi$ & S are both member of P(S).

eg - S ~~....~~ = $\{a, b\}$ then

$P(S) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$

Note :- If S be a finite set of order n the P(S) is finite set of order $2^n$.

eg - If $A = \{\phi, 1, \{b\}\}$

$P(S) = \{\phi, \{\phi\}, \{1\}, \{\{b\}\}, \{\phi, 1\}, \{\phi, \{b\}\}, \{1, \{b\}\}, \{\phi, 1, \{b\}\}\}$

Disjoint set ⇒

Two sets are said to be disjoint when they have no element in common. When no element of S in T and vice-versa.

eg - $S = \{a, b, c\}$, $T = \{e, f, g\}$

There is no any element are belongs to both set. Therefore these are called the disjoint.

→ · If S be the set of +ve number & T be the set of -ve no. then S & T are disjoint because no number is both +ve & -ve.

Comparable set ⇒

Two set S & T are said to be comparable if $S \subset T$ or $T \subset S$ that is if one of the set is the subset of other.

eg - $S = \{a, b, c\}$, $T = \{b, c\}$

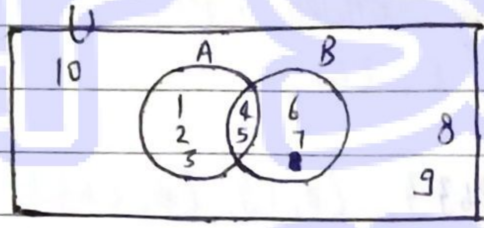
$\boxed{T \subset S}$

If one contains the other

Non-comparable set $\Rightarrow$ If $T \not\subset S$ & $S \not\subset T$
and if $S = \{a, b\}$ & $T = \{b, d, e\}$

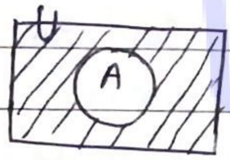
Here $b \in S$ but $b \in T$ and $d \notin S$ but $d \in T$
and $a \in S$ but $a \notin T$

Venn - Diagram $\Rightarrow$ Pictorial representation of set by area within circle.

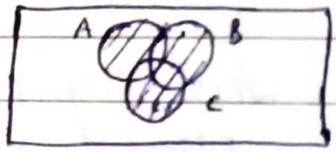



$A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$
$A \cap B = \{4, 5\}$
$A - B = \{1, 2, 3\}$
$B - A = \{6, 7\}$
$U' = \phi$
$(A \cup B)' = \{8, 9, 10\}$

Q. $U - A = A^c$




Q. $A \cup B \cup C$




Properties of Union operation:—

(i) $A \cup A = A$ (ii) $A \cup \phi = A$
(iii) $A \cup U = U$ (iv) $A \cup B = B \cup A$ (commutative)
(V) $(A \cup B) \cup C = A \cup (B \cup C)$ (associative)

proof – (V) To prove it

$(A \cup B) \cup C \subseteq A \cup (B \cup C)$ — (1)
& $A \cup (B \cup C) \subseteq (A \cup B) \cup C$ — (2)

Let $x \in (A \cup B) \cup C$
$x \in (A \cup B)$ and $x \in C$
$x \in A$ and $B$ and $x \in C$
$x \in A$ and $x \in B \cup C$
$x \in A \cup (B \cup C)$ . — (1)

Let $x \in A \cup (B \cup C)$

$x \in A$ and $x \in (B \cup C)$

$x \in A$ and $x \in B$ and $C$

$x \in (A \cup B)$ and $x \in C$

$x \in (A \cup B) \cup C$ — ② (H.P.)

Properties of intersection operation:—

(i) $A \cap A = A$ (ii) $A \cap \emptyset = \emptyset$

(iii) $A \cap U = A$ (iv) $A \cap B = B \cap A$

(v) $(A \cap B) \cap C = A \cap (B \cap C)$

Now Distriubution properties:—

(i) $A \cup (B \cap C) = \cancel{A} (A \cup B) \cap (A \cup C)$

proof:- To prove it

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ — ①

& $(A \cup B) \cap (A \cup C) = A \cup (B \cap C)$ — ②

Let $x \in A \cup (B \cap C)$

$x \in A$ or $x \in B$ and $C$

$x \in A$ or $\{x \in B$ and $x \in C)$

$\{x \in A$ or $x \in B\}$ and $\{x \in A$ or $x \in C\}$

$x \in (A$ or $B)$ and $x \in (A$ or $C)$

$x \in (A \cup B)$ and $x \in (A \cup C)$

$x \in (A \cup B) \cap (A \cup C)$

Therefore

$A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$ — ①

Now

let $x \in (A \cup B) \cap (A \cup C)$

$x \in (A$ or $B)$ and $(A$ or $C)$

$(x \in A$ or $x \in B)$ and $(x \in A$ or $x \in C)$

$x \in A$ or $\{x \in B$ and $x \in C\}$

$x \in A \cup \{x \in B \cap x \in C)$

$x \in A \cup (B \cap C)$

Therefore

$(A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$ — ② (H.P.)

(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

## Principle of Inclusion & Exclusion:—

(i) If set A & B are disjoint finite set then

$$|A \cup B| = |A| + |B|$$

(ii) If set A & B are finite set then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

(iii) If set A, B & C are finite set then

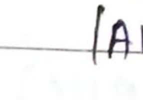
$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Partition :— A sets {A, B, C - - -} are ~~known~~ not empty sub set of S. The set S

(i) $A \cup B \cup C \text{ - - -} = S$

(ii) The intersectional subsets always $= \phi$

i.e. $A \cap B = \phi$, $B \cap C = \phi$, $A \cap C = \phi$ - - - so on




Q. A computer company must hire 25 programmer to handle system progr. job & 40 for the application programming for the hired profession. 10 will have to do the job of both type find how many programming must be hired.

Ay-

$n(SP) = 25$

$n(AP) = 40$

$n(SP \cap AP) = 10$


$n(SP \cap AP) = 10$

$n(SP \cup AP) = 25 + 40 - 10 = 55$ Ay

By the inclusion & exclusion principle Answer is 55.

Q. In a class containing ~~●~~ 50 students. 15 play tennis, 20 play ~~cricket~~ football and 20 play hockey, 6 play ~~cricket~~ football & hockey. 5 play tennis & hockey. 3 play tennis & football. 7 play no game. How many play football, tennis & hockey.

Ay-

By inclusion & Exclusion! —

$n(U) = 50$

$n(TP) = 15$, $n(FP) = 20$, $n(HP) = 20$

$n(F \cap H) = 6$, $n(T \cap H) = 5$, $n(T \cap F) = 3$

~~$n(F \cap H \cap T) = 7$~~ $n(F \cup H \cup T)' = 7$

~~$n(F \cup H \cup T) = 15 + 20 + 20 - 6 - 5 - 3 + 7$~~

~~$= 55 - 14 + 7 = 56 - 7 = 48$~~

$n(F \cup H \cup T) = n(U) - n(F \cup H \cup T)'$

$= 50 - 7 = 43$

$n(F \cap H \cap T) = 5 + 6 + 3 - 15 - 20 - 20 + 43$

$= 14 - 55 + 43 = 57 - 55$

$\boxed{n(F \cap H \cap T) = 2}$

Q. Let 100 of 120 students of mathematics at a college. Take at least one of the language Hindi, English & German also Let 65 study Hindi, 45 study English & 42 German. If 20 study H&E, 25 study E&G and 15 study H&G then find no. of students who read all the 3 language.

Ans-

$n(HP) = 65$, $n(EP) = 45$, $n(GP) = 42$
$n(H\&E) = 20$, $n(E\&G) = 25$, $n(H\&G) = 15$
$n(H \cap E \cap G) = ?$

$n(H \cup E \cup G) = 100$

$100 = 65 + 45 + 42 - 45 - 15 + n(H \cap E \cap G)$
$110 + 42 - 60 + n(H \cap E \cap G)$
$100 - 152 + 60 = n(H \cap E \cap G)$
$\boxed{8 = n(H \cap E \cap G)}$ Ans

Q. Prove that the following (Do not use example)

(i) $\overline{A \cup (B \cap C)} = (\bar{C} \cup \bar{B}) \cap \bar{A}$
(ii) $A \oplus B = (A \cup B) - (A \cap B)$

Ans- (i) $(A \cup (B \cap C))' = (\bar{C} \cup \bar{B}) \cap \bar{A}$ — (1)
& $(\bar{C} \cup \bar{B}) \cap \bar{A} = (A \cup (B \cap C))'$ — (2)

Let

$x \notin (A \cup (B \cap C))$
$x \notin A$ or $x \notin B$ and $x \notin C$

$x \notin A$ or $x \notin B$ and $x \notin A$ or $x \notin C$

$x \in A'$ or $x \in B'$ and $x \in A'$ or $x \in C'$

$x \in A'$ and $x \in B'$ or $x \in C'$

$x \in A' \cap x \in (B' \cup C')$

$x \in A' \cap (B' \cup C')$ — ① ~~similarly ②~~

~~(ii) $A \oplus B = (A \cup B) - (A \cap B)$~~

~~Ay - $A \oplus B = (A - B) \cup (B - A) = (A \cup B) - (A \cap B)$~~

Now to prove - ②

Let $y \in (\bar{C} \cup \bar{B}) \cap \bar{A}$

$y \in \bar{C}$ or $y \in \bar{B}$ and $y \in \bar{A}$

$y \notin C$ and $y \notin B$ or $y \notin A$

$y \notin C \cap B$ or $y \notin A$

$y \notin (C \cap B) \cup A$

$y \in \overline{(C \cap B) \cup A}$

$\bar{C} \cup \bar{B} \cap \bar{A} \subseteq \overline{(C \cap B) \cup A}$

Hence prove by ① & ② we can say the relation is true.

(ii) $A \oplus B = (A \cup B) - (A \cap B)$

Ay - $A \oplus B = (A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

$(A - B) \cup (B - A) \subseteq (A \cup B) - (A \cap B)$ — ①

& $(A \cup B) - (A \cap B) \subseteq A \oplus B$ — ②

Let

$x \in (A - B) \cup (B - A)$

$\{x \in A \cap x \notin B\}$ or $\{x \in B \cap x \notin A\}$

$\{x \in A$ and $x \notin B\}$ or $\{x \in B$ and $x \notin A\}$

$[x \in A$ or $x \in B]$ and $\{x \notin B$ or $x \notin A\}$

$(x \in A \cup B) \cap (x \in \bar{B}$ and $x \in \bar{A})$

$x \in (A \cup B) \cap (\bar{B} \cap \bar{A}) = x \in (A \cup B) - x \notin (B \cup A)$

PAGE NO

$x \in (A \cup B) - (B \cap A)$ ——— ①

Now to prove ②

$y \in (A \cup B) - (B \cap A)$

DATE : / /

PAGE NO. :

# Relation

## Order pair :-

Let a & b any two object . a is assigned as the first position and b as the second position . Then (a, b) is called the ordered pair of a & b .

Note → If a , b , c , d are 4 object then (a,b) = (c,d) & a = c and b = d unless a = b thus (a, b) ≠ (b, a)

Q. Find the no x & y is the ordered pair (2x-1, 5) and (x, y+1) are equal.

Ans - $(2x - 1, 5) = (x, y+1)$

then $2x - 1 = x$ , $5 = y+1$

$\boxed{x = 1}$ $\boxed{y = 4}$

## Cartisian product of sets ⇒

This is defined as

$A \times B = \{ (a, b) , a \in A , b \in B \}$ where $A \times B \neq B \times A$ unless $A = B$.

→ When , B = A then $A \times A = A^2$

Q. $A = \{1, 2, 3\}$ and $B = \{3, 4, 5, 6\}$ then find $A \times B$ & $B \times A$

Ans - $A \times B = \{ (1,3), (1,4), (1,5), (1,6), (2,3), (2,4), (2,5), (2,6), (3,3), (3,4), (3,5), (3,6) \}$

$B \times A = \{ (3,1), (3,2), (3,3), (4,1), (4,2), (4,3), (5,1), (5,2), (5,3), (6,1), (6,2), (6,3) \}$

where $(1,3) \in (A \times B)$ but $\notin (B \times A)$ H. $A \times B \neq B \times A$

Properties:—

(i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(iii) $A \times (B - C) = (A \times B) - (A \times C)$

(iv) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$



Proof:— (i) Let $(a, b) \in A \times (B \cup C)$

$a \in A \times b \in (B \cup C)$

$a \in A \times \{ b \in B \text{ or } b \in C \}$

$\{a \in A \times b \in B\}$ or $\{a \in A \times b \in C\}$

$(a, b) \in (A \times B) \cup (a, b) \in (A \times C)$

$(a, b) \in (A \times B) \cup (A \times C)$

(ii) Let $(a, b) \in A \times (B \cap C)$

$a \in A \times \{ b \in B \text{ and } b \in C \}$

$\{a \in A \times b \in B\}$ and $\{a \in A \times b \in C\}$

$(a, b) \in (A \times B) \cap (a, b) \in (A \times C)$

$(a, b) \in (A \times B) \cap (A \times C)$

(iii) $A \times (B - C) = (A \times B) - (A \times C)$

Let $(a, b) \in A \times (B - C)$

$a \in A \times \{ b \in (B - C) \}$

$a \in A \times \{ b \in B \cap b \notin C \}$

$(a \in A \times b \in B) \cap (a \in A \times b \notin C)$

$(a, b) \in (A \times B) \cap (a, b) \notin (A \times C)$

$(A \times B) - (A \times C)$

(iv) Let $(a,b) \in (A\times B) \cap (C\times D)$

$(a,b) \in (A\times B)$ and $(C\times D)$

$(a,b) \in (A\times B)$ and $(a,b) \in (C\times D)$

$(a \in A$ and $b \in B)$ and $a \in C$ and $b \in D$

$(a \in A$ and $a \in C)$ and $(b \in B$ and $b \in D)$

$(a \in (A\cap C))$ and $(b \in (B\cap D))$

$(a,b) \in (A\cap C)\times(B\cap D)$

Q. $A = \{2, 3, 5, 6\}$, $R \rightarrow$ divide. Represent set $R$

ARA

Ans- $R = \{(2,2), (2,6), (3,3), (3,6), (5,5), (6,6), (6,2), (6,3)\}$

Q. If $R$ be the set of real no., what is represented $R^2$ and $R^3$.

Ans-

$R^2 = \{(x,y) \mid x \in R, y \in R\}$

The element $x, y$ are the cordinate of the point in the cardison plane.

$R^3 = \{(x,y,z) \mid x \in R, y \in R, z \in R\}$

The coordinates of the point in 3 dimensional space.

## Domain and Range ⇒

Suppose $R$ is a relation from A to B i.e. $R \subseteq A\times B$ then the domain of $R$ = set of first element of the ordered pair.

Range of $R$ = Set of second element of the ordered pair.

## Inverse Relation :—

$$R^{-1} = \{ (y, x) \mid (x, y) \in R,\ x \in A,\ y \in B \}$$

Q. If $A = \{1, 2, 3, 4, 7\}$, $B = \{3, 4, 5\}$ $R \to$ defined $xRy \Leftrightarrow x < y$ then find the domain & Range.

Ans -

$A \times B = \{ (1,3), (1,4), (1,5), (2,3)\ (2,4)\ (2,5), \cancel{(3,3)}, (3,4), (3,5), \cancel{(4,3)}, \cancel{(4,4)}, (4,5) \}$

Domain $= \{1, 2, 3, 4\}$
Range $= \{3, 4, 5\}$

Q. $A = \{l, m, n\}$
$B = \{a, b\}$ then $R = \{(l, a), (l, b), (m, a), (m, b)\}$
$R \subseteq A \times B$
$A \Rightarrow B$
$R^{-1} = ?$

Ans -

$R^{-1} = \{ (a, l), (b, l), (a, m), (b, n) \}$

Range $= \{l, m, n\}$

→ R Relative set of an element $x$ :

$R(x) = \{ y \in B \mid (x, y) \in R \}$

Let R be a relation from A to B and $x \in A$ the R relative set of $x$ denoted by $R(x)$, is the set of those element of B which are related to that perticular element $x$.

Q. $A = \{1, 2, 3, 4\}$ , $B = \{r, s, t\}$
$R = \{(1, s)\ (1, t)\ (2, r)\ (2, s)\ (4, t)\}$
then find $R(1)$ $R(2)$ $R(4)$

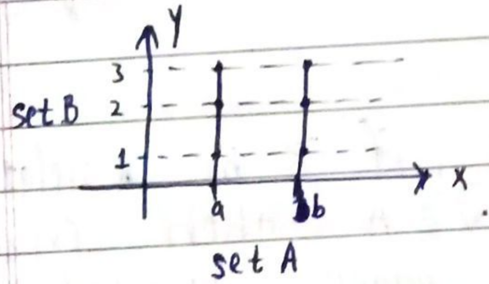
Sol?→ $R(1) = \{s, t\}$
$R(2) = \{r, s\}$
$R(4) = \{t\}$



## Graph of a Relation :—

Q. $A = \{a, b\}$ $B = \{1, 2, 3\}$
$R = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$

Ans –




## Adjacency Matrix of a Relation / Boolean matrix

⇒ $A = \{a_1, a_2, \dots a_m\}$
$B = \{b_1, b_2, \dots b_n\}$
$M_R = [M_{ij}]_{mn}$
where $M_{ij} = \begin{cases} 1, & \text{if } (a_i, b_j) \in R \\ 0, & \text{if } (a_i, b_j) \notin R \end{cases}$

The matrix $M_R$ is known as the matrix of Relation or Adjacency matrix or Boolean matrix of R

eg. $X = \{1, 2, 3\}$ $R = \{(2, 1), (3, 1), (3, 2)\}$
$M_R$ is $3 \times 2$ matrix

Sol?:– $M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}_{3 \times 2}$

DATE : / /
PAGE NO. :

Q. Find the relation R when is R relation from x to y where x

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}_{3\times5}$$

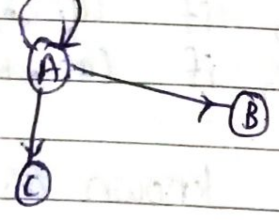
$R = \{(1,2), (2,1), (2,3), (2,4), (3,1), (3,3), (3,5)\}$

<u>Digraph</u> :- If R be the relation of an finite set ~~state~~ then R can be represented by pictorially. By an arc on edge. It is called <u>Digraph</u>. on <u>directed graph</u>.

→ <u>In degree of an vertex</u>, if R be a relation on a set A and $v \in A$ then indegree of v is the no. of edges directed thoward v.

→ <u>Out degree of a vertex</u>, The no. of edge beginning of a vertex that is going away from the v.

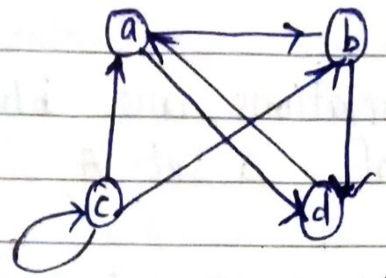
eg -



3 - out degree
1 → in degree

Q. $A = \{a, b, c, d\}$
$R = \{(a,b), (a,d) (b,d) (c,a) (c,b) (c,c) (d,a)\}$
then represent by the digraph and hence find the indgree & outdegree.

Ans-




| | a | b | c | d |
|---|---|---|---|---|
| Indegree | 2 | 2 | 1 | 2 |
| Outdegree | 2 | 1 | 3 | 1 |

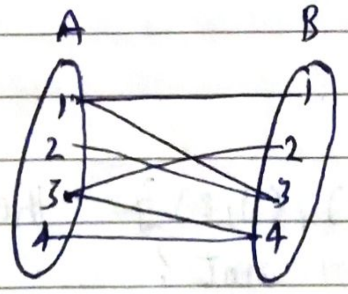
Q. If R are on the set $A = \{1, 2, 3, 4\}$ has the matrix representation $M_R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ Draw digraph & arrow diagram

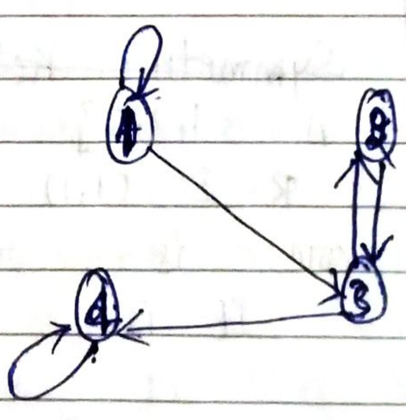
Ans-

$A = \{1, 2, 3, 4\}$

$M_R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$R = \{ (1,1), (1,3), (2,3), (3,2), (3,4), (4,4) \}$



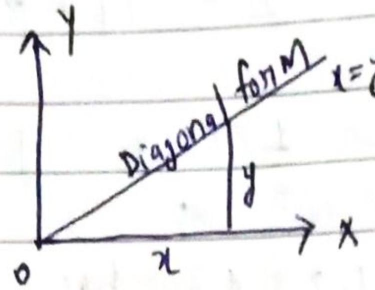




| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| Indegree | 1 | 1 | 2 | 2 |
| Outdegree | 2 | 1 | 2 | 1 |

Binary Relations :-

A relations are b/w a pairs of elements of a set A is called Binary Relation.

$\Delta = \{ (x,y) \mid x = y,\ x,y \in A \}$




Types of Binary relation

(i) Reflexive Relation :-

If R is a relation in the set A then R is called reflexive. If every element of A is a related to itself i.e. $(a,a) \in R \ \forall \ a \in A$

eg- (1,1), (2,2), (3,3)

~~(ii) Symmetric Relation :-~~

eg- A = {1, 2, 3}

R = { (1,1) (2,2) (3,3), (1,2) } then relation are is reflexive or not?

Ans - It is reflexive.

eg- If A = {1, 2, 3} and R = { (1,1) (2,2) (1,3) } that R is reflexive or not?

Ans - No reflexive. $(3,3) \notin R$

(ii) Symmetric :- If R is a relation then set A then $(a,b) \in R$ and $(b,a) \in R$

i.e. $\boxed{R = R^{-1}}$

eg- If A is a set $\{2,4,5,6\}$ and

(a) $R_1 = \{(2,4), (4,2), (4,5), (5,4), (6,6)\}$

(b) $R_2 = \{(2,4)\ (2,6), (6,2)\ (5,4), (4,5)\}$

check $R_1$ & $R_2$ are symmetric or not.

Ay. $R_1$ is symmetric

$R_2$ is not symmetric $(4,2) \notin R$

(iii) Transitive Relation :- If R is a relation in the set A then R is called transitive relation if $(a,b) \in R$ and $(b,c) \in R$ $\Rightarrow (a,c) \in R$

eg - If $A = \{1,3,5\}$

$R = \{(1,3)\ (1,5)\ (3,5)\}$

$(1,3) \in R$ , $(3,5) \in R$ then $(1,5) \in R$

Anti- symmetric ~~simuntaneously~~ :- R is a relation set A then R is called Anti- symmetric ~~simuntaneously~~ if $(a,b) \in R$ and $(b,a) \in R \Rightarrow a = b$

$a, b \in A$

Eg- In the set of sets the relation $\subseteq$ is Anti-symmetric since if A & B are any two sets then A is the subset of B and $B \subseteq A$ then $A = B$

Ay - This is Anti-symmetric.

$A \subseteq B$ $B \subseteq A$ $A = B$

Eg- In the set of natural no. the relation a divide b is anti-symmetric since $a|b$ and $b|a$ is possible only when $a=b$ i.e. if the given relation $(a,b) \in R$, $(b,a) \in R \Rightarrow a=b$

★ **Equivalence relation :—** If R is the relation in the set A, then R is called Equivalence relation (i) R is reflexive $(a,a) \in R, \forall a \in A$

(ii) R is symmetric $(a,b) \in R$, $(b,a) \in R$ where $a, b \in A$

(iii) R is transitive $(a,b) \in R$ and $(b,c) \in R \Rightarrow (a,c) \in R$ where $a, b, c \in R$

Q. Proof that in the set $A = \{1,2,3\}$ the relation are $R = \{(1,1)\ (2,2)\ (3,3)\ (2,1)\ (1,2)\ (2,3)\ (3,2)\ (3,1)\ (1,3)\}$

Ans-

(i) R is reflexive $(1,1), (2,2), (3,3) \in R$ $\forall\ 1,2,3 \in A$

(ii) R is Symmetric $(1,2) \in R$, $(2,1) \in R$
$(1,3) \in R$, $(3,1) \in R$
$(2,3) \in R$, $(3,2) \in R$

(iii) R is transitive $(1,2) \in R$ and $(2,3) \in R$ then $(1,3) \in R$
and $(2,1) \in R$ and $(1,3) \in R \Rightarrow (2,3) \in R$
so it is Equivalence relation.

Q. Consider a set $A = \{a, b, c, d, e, f\}$ and

$$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}_{6 \times 6}$$

Ay – $R = \{ (1,1)\ (1,2)\ (2,1)\ (2,2)\ (3,3)\ (4,4)\ (5,5)\ (6,6)$
$(4,5)\ (4,6)\ (5,4)\ (5,6)\ (6,4)\ (6,5) \}$

$R = \{ (a,a)\ (a,b)\ (b,a)\ (b,b)\ (c,c)\ (d,d)\ (e,e)\ (f,f)$
$(d,e)\ (d,f)\ (e,d)\ (e,f)\ (f,d)\ (f,e) \}$

(i) Reflexive :— $(a,a), (b,b), (c,c), (d,d), (e,e), (f,f) \in R$

(ii) Symmetric :— $(a,b), (b,a)\ (d,e), (e,d)$
$M_R = [M_R]^T$ $\quad (e,f)\ (f,e)\quad (d,f)\ (f,d)$

(iii) Transitive $\quad (a,b) \in R$ and $(b,a) \in R \Rightarrow (a,a) \in R$
$(d,e) \in R$ and $(e,d) \in R \Rightarrow (d,d) \in R$
$(d,f) \in R$ and $(f,f) \in R \Rightarrow (d,f) \in R$
$(e,f) \in R$ and $(f,e) \in R \Rightarrow (e,e) \in R$

or

$$M_R\, M_R = \begin{bmatrix} 1&1&0&0&0&0 \\ 1&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&1&1 \\ 0&0&0&1&1&1 \\ 0&0&0&1&1&1 \end{bmatrix} \begin{bmatrix} 1&1&0&0&0&0 \\ 1&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&1&1 \\ 0&0&0&1&1&1 \\ 0&0&0&1&1&1 \end{bmatrix} = \begin{bmatrix} 1&1&0&0&0&0 \\ 1&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&1&1 \\ 0&0&0&1&1&1 \\ 0&0&0&1&1&1 \end{bmatrix}$$

$\rightarrow 1 \oplus 1 \oplus 0 = 1$

## Equivalence Classes :—

$$[x] = \{ y / x ;\ y \in x,\ x R y \} = [y]$$

$$y \in [x] \Leftrightarrow x R y$$

If X be a set then R be a equivalence relation on x & x ∈ X then the set of all those member y ∈ X for which x relate y is called equivalence clause of x & it is denoted by ÷ ↑ Thus, an equivalence class is a subit of given set.

Properties of Equivalence Classes :—
Let set A be a non-empty set & R be an equivalence relation in A. Let x & y be an arbitrary element in A.

(i) $x \in [x]$
(ii) If $y \in [x]$ then $[y] = [x]$
(iii) $[x] = [y] \iff (x, y) \in R$ i.e. iff $xRy$
(iv) Either $[x] = [y]$ or $[x] \cap [y] = \phi$

Thus, two equivalence classes are either disjoint or identical.

**Q.** Let $A = \{1,2,3,4,5,6\}$ be the set & $R = \{(1,1)(1,5), (2,2)\ (2,3)\ (2,6)\ (3,2)\ (3,3)\ (3,6)\ (4,4)\ (5,1)\ (5,5), (6,2)\ (6,3)\ (6,6)\}$ be the equivalence relation of set & find the partition of A induced by A.

**Ans-**
$[1] = \{1, 5\}$
$[2] = \{2, 3, 6\}$
$[3] = \{2, 3, 6\}$
$[4] = \{4\}$
$[5] = \{1, 5\}$
$[6] = \{2, 3, 6\}$

i.e. $[1] = [5] = \{1, 5\}$
$[2] = [3] = [6] = \{2, 3, 6\}$
$[4] = \{4\}$

$\{[1], [2], [4]\} = \{\{1,5\}, \{2,3,6\}, \{4\}\}$

**Q** Consider A in set $A = \{1,2,3,4\}$, R is equivalence relation $R = \{(1,1)\ (1,2)\ (2,1)\ (2,2)\ (3,4)\ (4,3)\ (3,3)\ (4,4)\}$ find $A/R$ = partition of set A included by R

**Ans-**
$[1] = \{1, 2\}$
$[2] = \{1, 2\}$
$[3] = \{3, 4\}$
$[4] = \{3, 4\}$

i.e. $[1] = [2] = \{1, 2\}$
$[3] = [4] = \{3, 4\}$

$A/R = \{[1], [3]\} = \{\{1,2\}, \{3,4\}\}$

Composition of Relation :- Let R be a relation from B and S be a relation from B to C then, R and S give rise to a relation SoR from A to C which is called composition relation.

$SoR = \{(a,c) \mid (a,b) \in R \ \& \ (b,c) \in S\}$

$A = \{1,2,3,4\}$

$R = \{(1,2)\ (1,1)\ (1,3)\ (2,4)\ (3,2)\}$

$S = \{(1,4)(1,3)\ (2,3)\ (3,1)\ (4,1)\}$

$SoR = \{(1,3)\ (1,4)\ (1,1)\ (2,1)(3,1)\}$

Number of Partition of a finite set ⇒ Suppose N be a finite set of size n then, the number of partition of x (on the number of equivalence relation on x) is given by $\sum_{a=1} S(n,a)$ where $S(n,a)$ denotes a stirling number of second kind & it is defined by

$$\boxed{S(n,1) = 1 = S(n,n)}$$

and $\boxed{S(n,r) = S(n-1, r-1) + r\, S(n-1,r)}$

where $1 < r < n$

Q Compute the number of partitions of a set with:-

(i) four elements

(ii) five elements = 52

Ans (i) $S(4,1) = 1 = S(4,4)$ —— ①

$\sum_{r=1}^{n} S(n,r) = S(4,1) + S(4,2) + S(4,3) + S(4,4)$ —— ②

$S(4,2) = S(4-1, 2-1) + 2S(4-1, 2)$

$S(4,2) = S(3,1) + 2S(3,2)$ —— ③

$S(4,3) = S(3,2) + 3S(3,3)$ —— ④

where by the theorem

$$S(3,1) = 1 = S(3,3) \quad \text{—— } 5$$

and

$$S(3,2) = S(2,1) + 2 \cdot S(2,2)$$

$$S(2,1) = 1 = S(2,2) \quad \text{—— } 6$$

Now,

from - 1, 2, 3, 4, 5 & 6

$$= 1 + 7 + 6 + 1$$

$$= 15 \quad \text{Ay}$$

Partial Order Relation ⇒ A relation are defined on a set A said to be partial order relation in a if (i) R is reflexive
(ii) R is anti - symmetric
(iii) R is transitive

A set A together with define partial order relation are in A is known as partial ordered set or ~~c-s~~ poset, and defined by (A, R)

Eg- Let R be a relation in the set N of natural no. defined by "x divides y" then N is a partially ordered set with the relation R

Total order Relation →
/ Near order Relation A relation R define on a set k a said to be total / Near order relation in a if
(i) R is a partial order relation in A
(ii) Every pair of element of A is comparable with respect to R. i.e. for all the value of $a, b \in A$ either $aRb$ or $bRa$ or $a = b$

The element a & b of poset $(A, \leq)$ are said to be comparable if either $a \leq b$ or $b \leq a$.

HASSE Diagram ⇒
A partial ordering $(\leq)$ on a set P can be represented by a hasse diagram or a partially order set diagram $(P, \leq)$

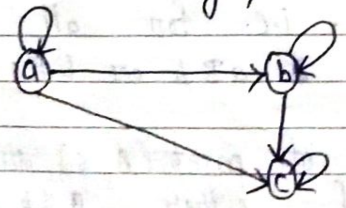
In this digraph, in the relation of the partial relation we follow the following step to make a simple.

(i) Delete all loop of the vertexes.
(ii) Delete all edges that must be present because of transitivity
(iii) Draw the digraph of the all partial order with all edge pointing upward in order to remove arrow from the edges.
(iv) Denote vertexes by dot in place of circle.

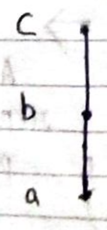
Hence, the digraph so obtained is called Hasse diagram of partial order relation.

Q. Draw the general & hasse diagram of the poset (A, R) where A = {a, b, c} and R = { (a,a), (b,b), (c,c), (a,b), (b,c), (a,c) }

Ans- General digraph:—



Hasse Diagram:—



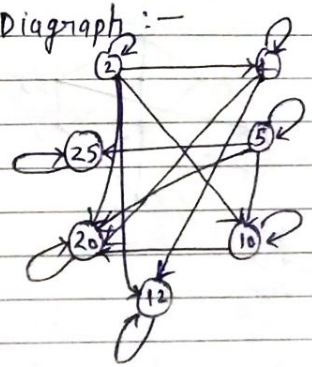
Q. Draw the hasse diagram
{ { 2, 4, 5, 10, 12, 20, 25 }, / }

Ans-

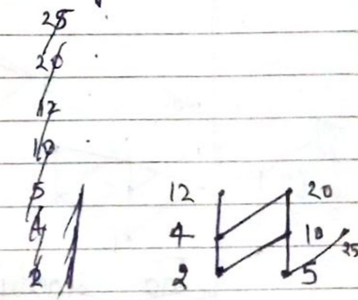
A×B ≠ B×A

R = { (2,2), (2,4), (2,10), (2,12), (2,20), (4,4) (4,12), (4,20), (5,10), (5,5), (5,20), (5,25), (10,10), (10,20), (12,12), (20,20), (25,25) }

Diagraph:—



Hasse diagram:—

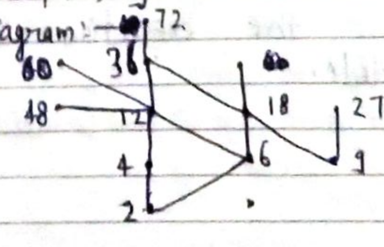


Q. Answer the following Question
{ {2, 4, 6, 9, 12, 18, 27, 36, 48, 60, 72}, / }

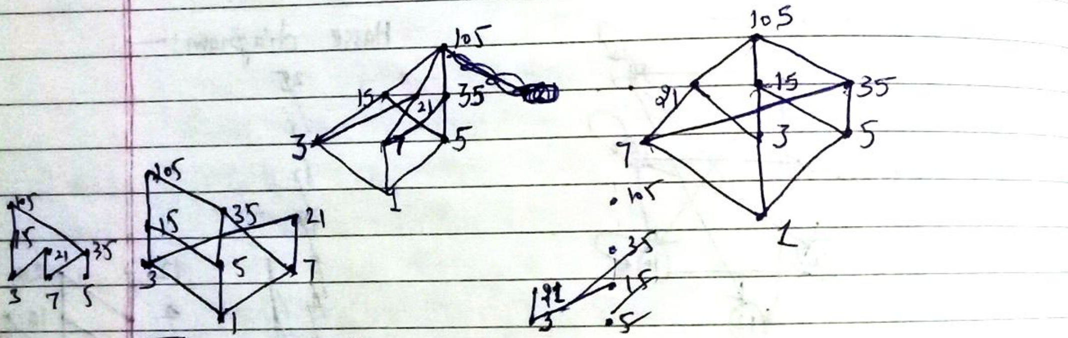
Ans -

R = { (2,2) (2,4) (2,6) (2,12) (2,18) (2,36) (2,48) (2,60) (2,72) (4,4) (4,12) (4,36), (4,48), (4,60), (4,72), (6,6) (6,12) (6,18) (6,36) (6,48) (6,60) (6,72) (9,9) (9,18) (9,27) (9,36) (9,72) (12,12) (12,36) (12,48) (12,60) (12,72) (18,18) (18,36) (18,72) (27,27) (36,36) (36,72) (48,48) (60,60) (72,72) }

Hasse diagram:—



Q. $X = \{1, 3, 5, 7, 15, 21, 35, 105\}$ & R be the relation '/' (divide) draw Hasse diagram.
(1,1) (1,3) (1,5) (1,7) (1,15) (1,21) (1,35) (1,105)
(3,3) (3,15) (3,21) (3,105) (5,5) (5,15) (5,35) (5,105)
(7,7) (7,21) (7,35) (7,105) (15,15) (15,105) (21,21)
(21,105) (35,35) (35,105) (105,105)



## Job scheduling ⇒

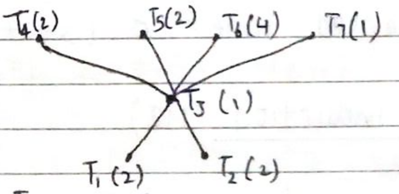
Let $T = \{T_1, T_2, T_3 - - - T_m\}$ denote a set of task to be executed on and identical processors.
Let $P_1, P_2, P_3 \cdot - - - - P_n$ these are the identical processor in a multiprocessor computing system.

→ Idle period is a time interval in which no task is executed on a certain processor called the idle period. It is denoted by $\phi$ or $\psi$.

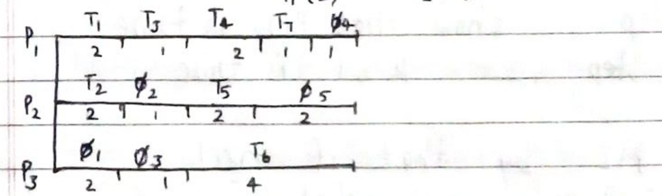
→ Total Elapsed time :- Total time required to complete the execution of all task as per the schedule.

Q. Consider a set of task $T = \{T_1 —— T_7\}$. A set of processor $P = \{P_1, P_2, P_3\}$ and a partial ordered relation ('≤') on T havings the pair $(T_1, T_3), (T_2, T_3), (T_3, T_4), (T_3, T_5), (T_3, T_6), (T_3, T_7)$. S its element other than the pair due to reflexing • T. ~~T~~ Execution time for $T_1$ is 2, $T_2$ is 2, $T_3$ is 1, $T_4$ is 2, $T_5$ is 2, $T_6$ is 4, $T_7$ is 1.
(i) describe problem graphically.
(ii) Draw the timing diagram.

Ans - (i)




(ii)

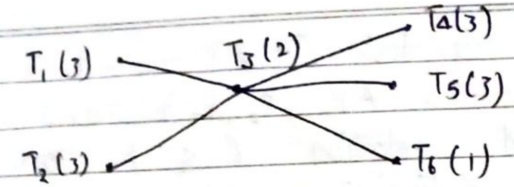


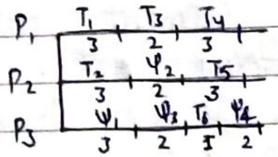

Total Idle time $= \phi_1 + \phi_2 + \phi_3 + \phi_4 + \phi_5$
$= 2 + 1 + 1 + 1 + 2 = 7$

Total ~~Ech~~ Elapsed time = 7

Q. In the following figure a set $\{T_1 T_2 — T_6\}$ of task is given. Draw a timing diagram. Let $\psi_1, \psi_2, \psi_3 - - - \psi_n$ be the Idle period of the processor.

PAGE NO. :




Total idle time $= \varphi_1 + \varphi_2 + \varphi_3 + \varphi_4$
$= 3 + 2 + 2 + 2 = 9$

Total Elapsed time $= 8$ $(3+2+3)$

Mathematical Induction (MI)

(i) Basic step, show that $P(1)$ is true
(ii) Inductive step, $k \geq 1$ is true

Q. Show by MI by that if $n \geq 1$
$1+2+3+ \cdots + n = \frac{n(n+1)}{2}$

Sol- Let $P(n) = 1+2+3+ \cdots + n = \frac{n(n+1)}{2}$

Basic step at $P(1) = 1 = \frac{1(1+1)}{2} = 1$ which is true

Inductive step at $P(k) = 1+2+3+ \cdots k = \frac{k(k+1)}{2}$ is also true

then
$P(k+1) = 1+2+3+ \cdots + k+1 = \frac{(k+1)(k+2)}{2}$

DATE : / /
PAGE NO. :

LHS of $P(k+1)$
i.e. $= \underbrace{1+2+3+ \cdots +k} + k+1$

$= \frac{k(k+1)}{2} + k+1$

$= \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)\{(k+1)+1\}}{2}$

$= \frac{(k+1)(k+2)}{2}$ = RHS (HP)

Q. Prove that the sum $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

Sol- Let
$P(n) = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

Basic step at $P(1) = 1 = \frac{1(1+1)(2+1)}{6} = 1$

Inductive step at $P(k) = 1^2 + 2^2 + \cdots k^2 = \frac{k(k+1)(2k+1)}{6}$

then we have to prove that
$P(k+1) = 1^2 + 2^2 + \cdots (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$

LHS of $P(k+1) = \underbrace{1^2 + 2^2 + \cdots + k^2} + (k+1)^2$

$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$

$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$

$= \frac{(k+1)\{k(2k+1) + 6k+6\}}{6}$

$= \frac{(k+1)\{2k^2 + 7k + 6\}}{6}$

$$\frac{(k+1)\{(k+1)+1\}\{2(k+1)+1\}}{6}$$

$= RHS$ (HP)

Q. Show that for all $n \geq 1$ i.e. $n! \geq 2^{n-1}$

Ans- let $P(n) = n! \geq 2^{n-1}$

<u>Basic step</u> at $P(1) = 1! \geq 2^{1-1} = 1 \geq 1$ (true)

<u>Inductive step</u> at $P(k) = k! \geq 2^{k-1}$ it also true then prove that

$P(k+1) = (k+1)! \geq 2^{k+1-1}$

LHS

$\underset{\geq 1}{(k+1)!} = (k+1)\,k! \geq (k+1)\,2^{k-1}$

then $k \geq 1$

$k+1 \geq 2$

so $\geq 2 \cdot 2^{k-1}$

$= 2^{k-1+1}$ $= RHS$ (H.P.)

# Pigeonhole principle / Dirichlet drawer principle / shoe base argument

If the no. of pigeon (i.e. object) is more than the no. of pigeonhole (i.e. boxes) then some pigeonhole must be occupied by two or more than 2 pigeon

eg- Let A & B are two finite sets such that the

$|A| > |B|$

then any function $f: A \rightarrow B$ can never be one In other words, their exist atleast two element

$a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$

<u>THEOREM-1</u> If n pigeons are assign to m pigeonholes where $m < n$ then atleast one pigeon hole contain two or more pigeon.

<u>THEOREM-2</u> <u>Generalise pigeonhole principle</u>

If n pigeons are assign to m pigeon hole where $m < n$ then some pigeonhole must contain at least $\left[\frac{n-1}{m}\right] + 1$ pigeons

$[k] \rightarrow$ denote the largest integer not greater than k. $(k+1)$

Q. Let there are 5 separate department in a departmental store & total no. of employees are 36. Show that one of the department must have atleast 8 employee.

Ans- $m = 5$

$n = 36$

therefore by the generalise pigeonhole principle one pigeonhole must contain $\left[\frac{36-1}{5}\right] + 1 = 8$ employees.

$\boxed{\text{Ans} - 8}$

Q. How many persons must be chosen in order that atleast 7 will have bdy in same calendar month?

Ans– Let the no. of req. person be n. We know that no. of months in which the bdys are distributed is 12. The pigeon hole principle, the least no. of persons who have their bday with same month is $\Rightarrow \left[\frac{n-1}{m}\right]+1=7$

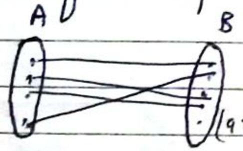
$$\frac{n-1}{12}+1=7 \Rightarrow n-1=6\times12$$
$$n=72+1 \Rightarrow \boxed{n=73}$$

## Functions

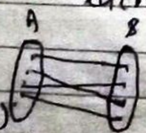
$f: A\to B$ or $A \xrightarrow{f} B$

Type of functions

(i) Into functions – If the function $f: A\to B$, at least one element of B is not the f image of any element of A.

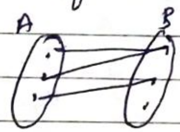



(ii) Onto functions – The mapping $f: A\to B$ is such that each elements of B has its image in A. (Also k/a subjective function)




(iii) One-One or Injective function:– A function $f: A\to B$ in which different element of A have different f image in B.


(iv) Many-one function:– If 2 or more element of A have same image in B.

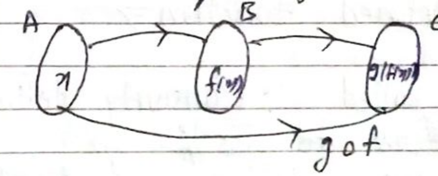



(v) One-One onto function (bijective):– If a function is one-one as well as onto, then $f: A\to B$ is bijective

$f(A)=f(B) \Rightarrow a=b$

$f(A)=B$

(vi) Composite function $\Rightarrow$ Let $f: A\to B$ and $g: B\to C$ then the composite of the function f & g is denoted by gof or gf is mapping $(gof)(x)=g[f(x)]\ \forall\ x\in A$




Properties of composite function

(i) Not Commutative

(ii) Associative

(iii) Composite of 2 bijections is a bijections.

Q. If $f(x)=x^2$, $g(x)=1/x$; then prove that $fog(1/2)$ and $gof(1/2)$ are equal.

DATE: / /
PAGE NO.:

$fog(x) = f[g(x)] = f(\frac{1}{x}) = \frac{1}{x^2}$

$gof(x) = g[f(x)] = g[x^2] = \frac{1}{x^2}$

Now, $fog(\frac{1}{2}) = f(\frac{1}{1/2}) = f(2) = (2)^2 = 4$

$gof(\frac{1}{2}) = g[\frac{1}{4}] = g = \frac{1}{1/4} = 4$

So $fog(\frac{1}{2}) = gof(\frac{1}{2})$ H.P.

Q. Obtain $fog$ & $gof$ :—

$f(x) = \sqrt{1-x^2}$, $g(x) = \log(1/x)$

$fog(x) = f[g(x)] = f(\log(\frac{1}{x}))$

$= \sqrt{1-2\log\frac{1}{x}}$

$gof(x) = g[f(x)] = g(\sqrt{1-x^2}) = \log(\frac{1}{\sqrt{1-x^2}})$

Recursively Defined function ⇒

A function is called recursively defined of definition of func^n refers to itself.

For this the definition of func^n should satisfy 2 conditions:—

(i) There must be certain arguments for which the func^n does not refer to itself, called ~~lease~~ values.

(ii) Each time, the func^n does not refers to itself, the argument of the function must be closer to a lease value.

DATE: / /
PAGE NO.:

Q. If $a$ & $b$ are +ve integers & function $g$ is defined recursively follows:—

$$g(a,b) = \begin{cases} 0 & , \text{if } a<b \\ g(a-b, b)+1 & , \text{if } a \geq b \end{cases}$$

then find (i) $g(3,5)$ (ii) $g(13,5)$ (iii) $g(9100,11)$

Ans - (i) $g(3,5) = 0$ $\quad 3<5$

(ii) $g(13,5) = g(13-5, 5)+1 = g(8,5)+1$

$= g(8-5, 5)+1+1 = g(3,5)+2$

$= 0+2 = 2$ Ans

(iii) $g(9100, 11)$ :— We observe there each time b is subtracted from a and the value of g is increased by 1. Hence $g(a,b)$ gives the quotient when $a \div b$ therefore $g(9100, 11)$ = quotient when $9100 \div 11 = 827$ is the answer.

Q. If func^n $n \to$ +ve integer and a func^n $g$ is defined recursively as follows:—

$$g(n) = \begin{cases} 0 & , \text{if } n=1 \\ g[\frac{n}{2}]+1 & \text{if } n>1 \end{cases}$$

find (i) $g(27)$ (ii) $g(32)$

Ans - $g(27) = g(\frac{27}{2})+1 = g[13]+1 = g[\frac{13}{2}]+1+1$

$g[6]+2 = g[\frac{6}{2}]+2+1 = g[3]+3$

$g[\frac{3}{2}]+3+1 = g[1]+4 = 0+4 = 4$ Ans

(ii) $g(32) = g(32/2)+1 = g(16)+1 \to g(16/2)+2$

$g(8)+2 = g(8/2)+3 = g(4)+3$

$g(4/2)+4 = g(2)+4 = g(2/2)+4+1 = 0+5$

$= 5$ Ans

**ACKERMAN FUNCTION** ⇒ It is a famous example of recursive function with 2 arguments and its determined by the following eq^n as follows:—

(i) $A(0,n) = n+1$ , for $n = 0,1,2 \text{-----}$

(ii) $A(m,0) = A(m-1, 1)$ , for $m = 1,2,3 \text{-----}$

(iii) $A(m,n) = A(m-1, A(m,n-1))$; for $m = 1,2,3 \text{-----}$, $n = 1,2,3 \text{-----}$

~~Ackermann~~

Q. Find $A(0,0)$, $A(0,1)$, $A(1,1)$, $A(1,2)$, $A(2,1)$, $A(2,2)$, $A(1,3)$

Sol^n:-
$A(0,0) = 0+1 = 1$

$A(0,1) = 1+1 = 2$ by (i)

$A(1,1) = A(1-1, A(1,1-1))$ by (iii)
$= A(0, A(1,0))$
$= A(0, A(1-1, 1))$ by (ii)
$= A(0, A(0,1))$
by (i)
$= A(0,2) = 2+1 = 3$

$A(1,2) = A(1-1, A(1, 2-1))$
$A(0, A(1,1))$
$= A(0, 3)$
$=$
$= 4$

$A(2,1) = 5$ , $A(2,2) = 7$ , $A(1,3) = 5$

$A(2,1) = A(1, A(2,0))$
$= A(1, A(1,1))$ $A(1,3)$

$A(2,1) = A(1,3) = A(1-1, A(1,3-1))$
$= A(0, A(1,2))$
$= A(0,4) = 4+1 = 5$

$A(2,2) = (A(2-1, A(2,2-1))$
$= A(1, A(2,1))$
$= A(1,5) = A(1-1, A(1,5-1))$
$= A(0, A(1,4))$
$= A(0, A(1-1, A(1,3)))$
$= A(0, A(0, A(1,3)))$
$= A(0, \{A(0,5)\})$
$= A(0,6) = 7$

$A(1,3) = A(1-1, A(1,2)) = A(0, A(1-1, A(1,1)))$
$= A(0, A(0,3))$
$= A(0,4) = 5$ Ans

Hasse Diagram:—

(i) Greatest element ⇒ An element $(A, \le)$ is said to be greatest if B is less equal to a $(b \le a)$ $\forall$ $b \in A$. The greatest element is unique when it exists.

(ii) Least element ⇒ An element $(A, \le)$ is said to be least if $a \le b$ $\forall$ $b \in A$. Since the least element is unique when it exists.

DATE: / /
PAGE NO.:

(iii) ~~Upperbond~~ Upperbound ⇒ Let $(A, \leq)$ and $a, b \in A$ then an element $c \in A$ is called upperbound of $a, b$ if $a \leq c$ and $b \leq c$.

(iv) Lowerbound ⇒ Let $(A, \leq)$ and $a, b \in A$ then an element $d \in A$ is called lowerbound of $a, b$ if $d \leq a$ and $d \leq b$

(v) Least upperbound:– An element $l \in A$ is called ~~these~~ Least upperbound of $a$ & $b$ in set A if
(i) $a \leq l$ and $b \leq l$
(ii) If there exists an element $l' \in A$ subject to $a \leq l'$ & $b \leq l'$ i.e. if $l$ is another upperbound of $a$ & $b$ then $l'$ is also the upperbound of $l$ which follow $\boxed{l \leq l'}$

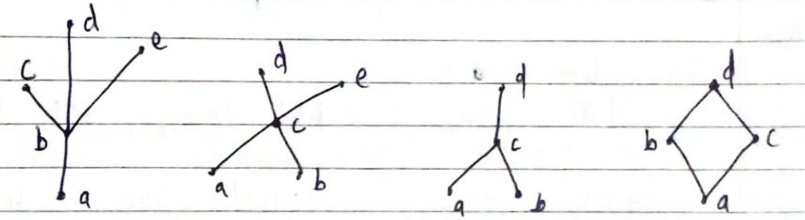
(vi) Greatest lowerbound (Infimum):– An element $g \in A$ is called infimum of $a$ & $b$ in set A if
(a) $g \leq a$ and $g \leq b$
(b) if there exist an element $g' \in A$ s.t. $g' \leq a$ and $g' \leq b$ then $\boxed{g' \leq g}$.

(vii) Maximal elements:– If there is no element $b \in A$ such that $a < b$ (top element in a hasse diagram)

DATE: / /
PAGE NO.:

(viii) Minimal element:– If there is no element $b \in A$ such that $b < a$ (bottom element in a hasse diagram)

Q. Find the greatest element & lowest element

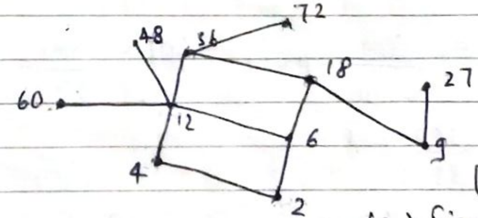



(i) (ii) (iii) (iv)

Ans– (i) Greatest element don't exists.
(ii)

| | (i) | (ii) | (iii) | (iv) |
|---|---|---|---|---|
| greatest | DNE | DNE | d | d |
| least | a | DNE | DNE | a |

Q.




(a) find minimal & maximal elements
bottom element
top element
(b) Greatest & least element
(c) Find lub of {2, 9} if exist
(d) GLB of {60, 72} if exist

Ans– (a) maximal – 27, 48, 60, 72
minimal – 2, 9
(b) Greatest element – 72
Lower element – 2

DATE: / /
PAGE NO.:

(c) LUB :— 18 $\qquad$ UB → 18, 36, 72

GLB :— 12 $\qquad$ LB → 12, 4, 6, 2

Q. 1 to 300 integers find

(i) How many of them are not divided by 3, 5, 7.

(ii) How many of them are divided 3 and not dividual by 5, 7.

Ay-

(i) $U = 300$

$|A| = \frac{300}{3} = 100$, $\quad |B| = \frac{300}{5} = 60$, $\quad |C| = \frac{300}{7} = 42$

$|A \cap B| = \frac{300}{15} = 20$, $\quad |B \cap C| = \frac{300}{35} = \frac{60}{7} = 8$

$|A \cap C| = \frac{300}{21} = \frac{100}{7} = 14$, $\quad |A \cap B \cap C| = \frac{300}{105} = 2$

$|A \cup B \cup C| = 100 + 60 + 42 - 20 - 8 - 14 + 2$

$|A \cup B \cup C| = 202 - 40 = 162 \Rightarrow 300 - 162 = \underline{138}$

(ii) $|A \cup B| = |A| + |B| - |A \cap B|$

$|A| = \frac{300}{5} = 60$, $\quad |B| = \frac{300}{7} = 42$

$|A \cap B| = \frac{300}{35} = 8$

$|A \cup B| = 60 + 42 - 8 = 102 - 8 = 94$

$= |A \cup B \cup C| - |A \cup B|$

$= 162 - 94 = \underline{68}$ Ay

DATE: / /
PAGE NO.:

Q. Find the no. of the integer which is less than equal to 3000 and not division by 7 & 8.

Ay- $U = 3000$

$|A| = \frac{3000}{7} = 425$ $\qquad |A \cap B| = \frac{3000}{56} = \frac{425}{8} = 53$

$|B| = \frac{3000}{8} = 375$

$|A \cup B| = 425 + 375 - 53$

$|A \cup B| = 800 - 53 = 747$

$|\overline{A \cup B}| = 3000 - 747 = \underline{2253}$ Ay

Q. Find the min. no. of student in the class to be sure that there 3 are born in the same month.

Ay-

~~$|A \cup B| = |A| + |B| - |A \cap B|$~~

$n = 12$,

$k + 1 = 3$ (Given)

$kn + 1$

$2 \times 12 + 1 = 25$

Q. 52 total students

16 - Drink tea

17 - Drink tea & coffee.

(i) how many students who drink coffee?

(ii) how many students who not drink coffee & tea both?

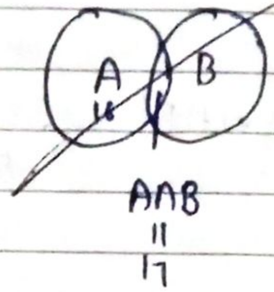
Ay- $U = 52$

$A = 16$ $\qquad$ (i) $B = ?$

$A \cap B = 17$ $\qquad$ (ii) $\overline{A \cap B} = ?$

$|A \cup B| = |A| + |B| - |A \cap B|$

$17 = 16 + |B| - 17$




$|B| = 17 - 16 = 1$

(ii) $52 - (16 + 17)$

$52 - 33 = 19$ Ans

Ans-(i) Let T = Tea

C = coffee

drink T but not C $n(T - C) = x_1$

drink T & C both $n(T \cap C) = x_2$

drink C but not T $n(C - T) = x_3$


$x_1 + x_2 + x_3 = 52$

$x_1 = 16$,

$\therefore x_1 + x_2 = 33$ $(16 + 17 = 33)$

$x_3 = 52 - 33 = 19$

(i) who drink coffee but not tea $= 19 = x_3$ Ans

(ii) who drink both $= 17 = x_2$ Ans

# Unit - 2
## Propositional logic

It is a declarative statement i.e. either true or false but not both. For true assignment we assign a value T and for false assignment we assign the value F. And these T & F are called truth values.

eg - (i) Delhi is in India (T)
(ii) 3+5 = 8 (T)
(iii) Jaipur is a state (F)
(iv) What are you doing? (It is question, not a state.)

Login connectivity & its compound statement

| Symbol | Connective | Name |
|---|---|---|
| $\sim$ or $\neg$ | not | negation |
| $\wedge$ | and | conjuction |
| $\vee$ | or | disjunction |
| $\longrightarrow$ | implies or if then | Implication / condi^n |
| $\longleftrightarrow$ | if and only if | Equivalence & bicondition |

**Atomic Prime statement** $\Rightarrow$

The statement or preposition which doesn't contents any connections is called the prime or atomic statement.

PAGE NO. :

1 Negetion

| $p$ | $\neg p$ |
|---|---|
| T | F |
| F | T |

2 Conjection:—

| $p$ | $q$ | $p \wedge q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

3 Disjunction:—

| $p$ | $q$ | $p \vee q$ |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

$3+4>8$ or jaipur is the capital of india.
Where $p \vee q$: False both $p$ & $q$ are false.

4 Implication or conditional:—

| $p$ | $q$ | $p \to q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

Eg – $p$: I am hungry
$q$: I will eat
$p \to q$: If i am hungry then i will eat.

DATE: / /
PAGE NO. :

Implication:— $p \to q$

Converse:— $q \to p$
Inverse:— $\neg p \to \neg q$
Contrapositive:— $\neg q \to \neg p$

| Biconditional $p \longleftrightarrow q$ | $p$ | $q$ | $p \to q$ | $q \to p$ | $(p \to q) \wedge (q \to p)$ |
|---|---|---|---|---|---|
| | T | T | T | T | T |
| | T | F | F | T | F |
| | F | T | T | F | F |
| | F | F | T | T | T |

Q. Find the converse, contrapositive & inverse of the following implication
Statement:— If today is thursday then I have a test today.

Ans – Let $p$: today is thursday and
$q$:— I have a test today

Then
Converse: $q \to p$
$q \to p$ If I have a test today then today is thursday

contrapositive: $\neg q \to \neg p$
$\neg q \to \neg p$ If I don't have a test today then tody is not thursday.

Inverse: $\neg p \to \neg q$
$\neg p \to \neg q$ If ~~the~~ today is not thursday then I don't have a any test today.

Truth table:—

| p | q | $p\to q$ | $q\to p$ | $\sim p$ | $\sim q$ | $\sim q\to\sim p$ | $\sim p\to\sim q$ |
|---|---|---|---|---|---|---|---|
| T | T | T | T | F | F | T | T |
| T | F | F | T | F | T | F | T |
| F | T | T | F | T | F | T | F |
| T | T | T | T | F | F | T | T |

Q. Make truth table for

(i) $p \vee \sim q$ (ii) $(\sim p \wedge q) \vee p$

(iii) $(p \vee q) \vee \sim q$ (iv) $(p\vee q) \wedge R$

(v) $(\sim p \vee q) \wedge \sim r$ (vi) $\sim(\sim p)$

Ans –

| p | q | $\sim p$ | $\sim q$ | $p\vee\sim q$ | $\sim(\sim p)$ | $\sim(p\wedge q)\vee p$ | $(p\vee q)\vee\sim q$ |
|---|---|---|---|---|---|---|---|
| T | T | F | F | T | T | T | T |
| T | F | F | T | T | T | T | T |
| F | T | T | F | F | F | T | T |
| T | T | F | F | T | T | T | F |

| p | q | R | $(p\vee q)\wedge R$ | $\sim p\vee q$ | $(\sim p\vee q)\wedge\sim R$ |
|---|---|---|---|---|---|
| T | T | T | T | T | F |
| T | T | F | F | T | T |
| T | F | T | T | F | F |
| T | F | F | F | F | F |
| F | T | T | T | T | F |
| F | T | F | F | T | T |
| F | F | T | F | T | F |
| F | F | F | F | T | T |

Tautology :—

$$(p\to q) \longleftrightarrow (\sim q \to \sim p)$$

A compound statement that is always true for all possible truth values

Contradiction :—

or absurdity :— $(p \wedge \sim p) \equiv F$

A compound statement i.e. always false

Contingency :— $(p\to q) \wedge (p \vee q)$

If a statement i.e. neither a ~~topolo~~ tautology nor a contradiction is called contingency. So its truth value contain both T & F values at the last column.

Q. Show that the statement $(p\to q) \Longleftrightarrow (\sim q \to \sim p)$ is tautology and stat. $(p\wedge\sim p)\equiv F$ is contradiction and state. $(p\to q)\wedge(p\vee q)$ is contingency.

Ans –

| p | q | $\sim p$ | $p\to q$ | $\sim q$ | $\sim q\to\sim p$ | $(p\to q)\to(\sim q\to\sim p)\wedge(\sim q\to\sim p)\to(p\to q)$ |
|---|---|---|---|---|---|---|
| T | T | F | T | F | T | T |
| T | F | F | F | T | F | T |
| F | T | T | T | F | T | T |
| F | F | T | T | T | T | T |
| | | | (i) | | (ii) | |

from truth table $p\to q \Longleftrightarrow \sim q \to \sim p$ ✓

So it is tautology.

| $p$ | $\sim p$ | $(p \wedge \sim p)$ |
|---|---|---|
| T | F | F |
| F | T | F |

According to truth table $(p \wedge \sim p) \equiv F$ contradiction

| $p$ | $q$ | $p \to q$ | $p \vee q$ | $(p \to q) \wedge (p \vee q)$ |
|---|---|---|---|---|
| T | T | T | T | T |
| T | F | F | T | F |
| F | T | T | T | T |
| F | F | T | F | F |

According to truth table
$(p \to q) \wedge (p \vee q)$ is contingency.

## Logical Equivalent :—

2 compound state. $p$ & $q$ are said to be logical equ. if $p \longleftrightarrow q$ is tautology.
If $p \equiv q$ then we write $\boxed{p \equiv q}$

Q. Show that $p \vee q$ and $q \vee p$ are equivalent.

Ans-

| $p$ | $q$ | $p \vee q$ | $q \vee p$ | $p \vee q \to q \vee p$ | $q \vee p \to p \vee q$ | $(p \vee q) \leftrightarrow (q \vee p)$ |
|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T |
| T | F | T | T | T | T | T |
| F | T | T | T | T | T | T |
| F | F | F | F | T | T | T |

$\boxed{(p \vee q) \equiv (q \vee p)}$
$= (p \vee q) \to (q \vee p) \wedge (q \vee p \to p \vee q)$

**Note :-** If the 2 statement has ~~same~~ identical value then we can say these statement are equivalent.

RTU Q. Show that $p \to q \equiv (\sim p) \vee q$

Ans -

| $p$ | $q$ | $\sim p$ | $\sim p \vee q$ | $p \to q$ |
|---|---|---|---|---|
| T | T | F | T | T |
| T | F | F | F | F |
| F | T | T | T | T |
| F | F | T | T | T |

Hence this statements are equivalent.

Since $p \to q$ & $\sim p \vee q$ are identical so
$p \to q \equiv \sim p \vee q$

## Operation for Proposition

(A) Cummutative ⇒
(i) $p \vee q \equiv q \vee p$
(ii) $p \wedge q \equiv q \wedge p$

(B) Associative ⇒
(i) $p \vee (q \vee r) \equiv (p \vee q) \vee r$
(ii) $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$

(C) Distributive ⇒
(i) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
(ii) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

(D) Idempotent →
(i) $p \vee p \equiv p$
(ii) $p \wedge p \equiv p$

(e) Properties of Negation ⇒

(i) $\sim(\sim p) \equiv p$

(ii) $\sim(p \vee q) \equiv \sim p \wedge \sim q$ ] De-Morgan Law

(iii) $\sim(p \wedge q) \equiv \sim p \vee \sim q$

(f) Identify laws :⇒

(i) $p \wedge T \equiv p$ (ii) $p \vee F \equiv p$

(g) Domination Laws ⇒

(i) $p \vee T \equiv T$ (ii) $p \wedge F \equiv F$

(h) Absorption Law ⇒

(i) $p \vee (p \wedge q) \equiv p$

(ii) $p \wedge (p \vee q) \equiv p$

Additional Logical Equivalence

(i) $p \vee (\sim p) \equiv T$

(ii) $p \wedge (\sim p) \equiv F$

(iii) $(p \to q) \equiv (\sim p \vee q)$

(iv) $(p \leftrightarrow q) \equiv (p \to q) \wedge (q \to p)$

**Q.** Show that $\sim(p \vee (\sim p \wedge q)) \equiv (\sim p) \wedge (\sim q)$

**Ans-** LHS

$\sim(p \vee (\sim p \wedge q))$

$= \sim p \wedge \sim(\sim p \wedge q)$ (from de-morgan

$\sim p \wedge (p \vee \sim q)$ (from de-morgan

$(\sim p \wedge p) \vee (\sim p \wedge \sim q)$ (By distribution law

$= F \vee (\sim p \wedge \sim q)$ (By identify law

$(\sim p) \wedge (\sim q)$

(H.P.)

**Q.** Show that $(p \wedge q) \longrightarrow (p \vee q)$ is a tautology

**Ans-** As we know that $p \to q \equiv (\sim p) \vee q$

$\equiv \sim(p \wedge q) \vee (p \vee q)$

$\equiv \sim p \vee \sim q \vee (p \vee q)$

$\equiv (\sim p \vee p) \vee (\sim q \vee q)$

$\equiv T \vee T \equiv T$

• NOR, or Joint Denial ($\downarrow$) :—

$p \downarrow q$ neither p nor q

| p | q | $p \downarrow q$ | $\sim(p \vee q)$ |
|---|---|---|---|
| T | T | F | F |
| T | F | F | F |
| F | T | F | F |
| F | F | T | T |

~~$p \downarrow q \equiv \sim(p \wedge q)$~~

$p \downarrow q \equiv \sim(p \vee q)$

NAND ($\uparrow$) :—

negation of and

| p | q | $p \uparrow q$ | $\sim(p \wedge q)$ |
|---|---|---|---|
| T | T | F | F |
| T | F | T | T |
| F | T | T | T |
| F | F | T | T |

$p \uparrow q \equiv \sim(p \wedge q)$

XOR $\oplus$ :— p is true or q is true. But not both p & q be true.

| p | q | $p \oplus q$ |
|---|---|---|
| T | T | F |
| T | F | T |
| F | T | T |
| F | F | F |

Q. $p \uparrow q \uparrow r$, $p \downarrow q \downarrow r$, $p \oplus q \oplus r$

Ans-

| p | q | r | $p \uparrow q$ | $p \uparrow q \uparrow r$ | $p \downarrow q$ | $p \downarrow q \downarrow r$ | $p \oplus q$ | $p \oplus q \oplus r$ |
|---|---|---|---|---|---|---|---|---|
| T | T | T | F | T | F | F | F | T |
| T | T | F | F | T | F | T | F | T |
| T | F | T | T | F | F | F | T | T |
| T | F | F | T | T | F | T | T | T |
| F | T | T | T | F | F | F | T | T |
| F | T | F | T | T | F | T | T | T |
| F | F | T | T | F | T | F | F | T |
| F | F | F | T | T | T | F | F | F |
| | | | | (i) | | (ii) | | (iii) |

Predicates and Quantifiers ⇒

Let the statement 'x is a positive ~~number~~ integer' this statement can not be true value unless the value of the variable x is specified. The first part of statement i.e. x is called the subject of statement, while the second part 'is a positive integer' is a predicate.

Eg- Let p(x) denotes the statement "x > 3" then what are the true value of P(2) and P(4)

Ans- P(x): "x > 3"
P(2): 2 > 3 "F"
P(4): 4 > 3 "T"

Quantifier :- This is a powerful technique to create a statement from a proportional function. There are 2 type of Quantifier:-
(i) Universal [denoted by $\forall x\ P(x)$]
(ii) Existential [denoted by $\exists x\ P(x)$]

Table of Quantifier for one variable ⇒

| Statement | When True ? | When False ? |
|---|---|---|
| $\forall x\ P(x)$ | P(x) is true for every x | There is at least one x for which P(x) is false. |
| $\exists x\ P(x)$ | There is atleast one x for which P(x) is true | P(x) is false for every x. |

Q. Express statement: 'Everyone has exactly one best friend' as a logical expression using Quantifier.

Ans- Let Q(x, y) denotes y is the best friend of x. Now the given statement means that for each person x there is another unique person y such that y has the best friend of x. It means if z is a person other than y then z can not be a best friend of x. Thus we can translate the given statement as

$$\forall x\ \exists y\ \forall z\ (Q(x,y) \land ((z \neq y) \to \sim Q(x,z)))$$

RTU
Q. With the help of truth table prove that
(i) $\sim(P \to q) \equiv P \land \sim q$
(ii) $P \to q \equiv \sim q \to \sim P$

(iii) $p \rightarrow (q \wedge r) \equiv (p \rightarrow q) \wedge (p \rightarrow r)$

Ans- (i) $\neg(p \rightarrow q) \equiv p \wedge \neg q$

| $p$ | $q$ | $p \rightarrow q$ | $\neg(p \rightarrow q)$ | $\neg q$ | $p \wedge \neg q$ |
|---|---|---|---|---|---|
| T | T | T | F | F | F |
| T | F | F | T | T | T |
| F | T | T | F | F | F |
| F | F | T | F | T | F |

(ii) $p \rightarrow q \equiv \neg q \rightarrow \neg p$

| $p$ | $q$ | $\neg p$ | $\neg q$ | $p \rightarrow q$ | $\neg q \rightarrow \neg p$ |
|---|---|---|---|---|---|
| T | T | F | F | T | T |
| T | F | F | T | F | F |
| F | T | T | F | T | T |
| F | F | T | T | T | T |

(iii) $p \rightarrow (q \wedge r) \equiv (p \rightarrow q) \wedge (p \rightarrow r)$

| $p$ | $q$ | $r$ | $q \wedge r$ | $p \rightarrow (q \wedge r)$ | $p \rightarrow q$ | $p \rightarrow r$ | $(p \rightarrow q) \wedge (p \rightarrow r)$ |
|---|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T | T |
| T | T | F | F | F | T | F | F |
| T | F | T | F | F | F | T | F |
| T | F | F | F | F | F | F | F |
| F | T | T | T | T | T | T | T |
| F | T | F | F | T | T | T | T |
| F | F | T | F | T | T | T | T |
| F | F | F | F | T | T | T | T |

DATE : / /

PAGE NO. :

Q. Show that $(p \wedge q) \wedge \sim(p \vee q)$ is a contradiction or failacy

Ans -

| p | q | $p \wedge q$ | $p \vee q$ | $\sim(p \vee q)$ | $(p \wedge q) \wedge \sim(p \vee q)$ |
|---|---|---|---|---|---|
| T | T | T | T | F | F |
| T | F | F | T | F | F |
| F | T | F | T | F | F |
| F | F | F | F | T | F |

Q. Show that $p \wedge q \rightarrow p \vee q$ is a Tautology

Ans -

| p | q | $p \wedge q$ | $p \vee q$ | $p \wedge q \rightarrow p \vee q$ |
|---|---|---|---|---|
| T | T | T | T | T |
| T | F | F | T | T |
| F | T | F | T | T |
| F | F | F | F | T |

Er Sahil Ka Gyan

Q. Over the universe all animals let

$P(x)$ : x is a whale

$Q(x)$ : x is a fish

$R(x)$ : x lives in water

Now translate this in English statement

(i) $\exists x\ (\sim R(x))$

(ii) $\exists x\ (Q(x) \wedge \sim P(x))$

(iii) $\forall x\ (P(x) \wedge R(x)) \longrightarrow Q(x)$

Ans (i) There exists an animal which does not live in water.

(ii) There exists a fish i.e. not a whale.

(iii) Every whale that live in water is a fish.

Q. Write contrapositive, converse & inverse of the statement 'The home team wins whenever it is raining'. also construct the truth table for each table.

Ans-

The statement can be written as if it is raining then home team wins

P : It is raining

Q : Home team wins

S : $p \rightarrow q$

Contrapositive:— $\sim q \rightarrow \sim p$

It is not raining the home team does not win.

Converse:— $q \rightarrow p$

It is raining then home team wins

Inverse:— $\sim p \rightarrow \sim q$

The home team does not win whenever it is not raining.

Normal forms:—

① Use "Product" in place of "conjunction" ($\wedge$)

② Use "SUM" in place of "Disjunction" ($\vee$)

(i) Elementary product:— A product of variables & their negation is called the elementary product. $p \wedge q$, $\sim p \wedge q$, $p \wedge \sim q$, $\sim p \wedge \sim q$ etc

DATE : / /

PAGE NO. :

Elementry sum :- $P \vee Q$, $\neg P \vee Q$, $P \vee \neg Q$, $\neg P \vee \neg Q$

Factor :- $P * Q$, $P \wedge Q$.

Minterms :- let P & Q are the 2 propositional variable. ~~All~~ possible formula with consists of product of P of its negation ($\neg P$). ~~and P&Q~~ But should not contain both the variable and its negation in any one of the formula are called minterms of P&Q.

$P \wedge Q$, $P \wedge Q$, $P \wedge \neg Q$, $\neg P \wedge Q$ but not $P \wedge \neg P$, $Q \wedge \neg Q$

Note:- (i) Each minterm has the truth value T for exactly one combination of the truth values of the variable P & Q.

(ii) If there are n variable then total no. of $\boxed{\text{minterm} = 2^n}$

Maxterm :- $P \vee Q$, $P \vee \neg Q$, $\neg P \vee \neg Q$, $\neg P \vee Q$ but not $P \vee \neg P$, $Q \vee \neg Q$

$\boxed{\text{maxterm} = 2^n}$

Note:- Each maxterm has the truth value F for exactly one combination of the truth values of variable P & Q.

## NOTE

(i) $P \vee T \equiv T$ (ii) $P \wedge T \equiv P$
(iii) $P \vee F = P$ (iv) $P \wedge F \equiv F$
(v) $\sim P \wedge P \equiv F$ (vi) $\sim P \vee P \equiv T$

DNF (Disjunction Normal forms)
or
SOP (sum of product) form $\Rightarrow$ A statement which consists of 'sum of elementary product' of proposition variable is equivalent to given compound statement. This form is not unique for the given statement.

CNF (conjuctive Normal forms)
or
POS (Product of sum) form $\Rightarrow$ A stat. which consists of 'product of elementary sum'.

Q. Obtain DNF of statement.
$P \wedge (P \rightarrow Q)$

Ans -

$P \wedge (\sim P \vee Q)$
$(P \wedge \sim P) \vee (P \wedge Q)$ (From Distribution Law

Q. Obtain DNF of statement.
$\sim(P \vee Q) \leftrightarrow P \wedge Q$

Ans - $(\sim(P \vee Q) \rightarrow (P \wedge Q)) \wedge ((P \wedge Q) \rightarrow \sim(P \vee Q))$

$$((p \vee q) \vee (p \wedge q)) \wedge (\neg(p \wedge q) \vee \neg(p \vee q))$$

$$((p \vee q) \vee (p \wedge q)) \wedge (\neg(p \wedge q) \vee \neg(p \vee q))$$

$$[(p \vee q) \vee (p \wedge q) \wedge \neg(p \wedge q)] \vee [(p \vee q) \vee (p \wedge q) \wedge \neg(p \vee q)]$$

$$\cancel{(p \wedge \neg p) \vee (p \wedge \neg q) \vee (q \wedge \neg p) \vee (q \wedge \neg q)}$$

$$[(p \vee q) \vee (p \wedge q) \wedge (\neg p \vee \neg q)] \vee [(p \vee q) \vee (p \wedge q) \wedge (\neg p \wedge \neg q)]$$

$$[((p \vee q) \vee (p \wedge q) \wedge \neg p) \vee ((p \vee q) \vee (p \wedge q) \wedge \neg q)] \vee$$
$$[((p \vee q) \vee (p \wedge q) \wedge \neg p) \wedge ((p \vee q) \vee (p \wedge q) \wedge \neg q)]$$

---

$$\neg(p \vee q) \longleftrightarrow (p \wedge q)$$

$$(\neg(p \vee q) \rightarrow (p \wedge q)) \wedge ((p \wedge q) \rightarrow \neg(p \vee q))$$

$$(\neg\neg(p \vee q) \vee (p \wedge q)) \wedge (\neg(p \wedge q) \vee \neg(p \vee q))$$

$$((p \vee q) \vee (p \wedge q)) \wedge (\neg(p \wedge q) \vee \neg(p \vee q))$$

$$[((p \vee q) \vee (p \wedge q)) \wedge \neg(p \wedge q)] \vee [((p \vee q) \vee (p \wedge q)) \wedge \neg(p \vee q)]$$

By appling distribution

$$[((p \vee q) \wedge \neg(p \wedge q)) \vee (p \wedge q) \wedge \neg(p \wedge q)] \vee [((p \vee q) \wedge \neg(p \vee q)) \vee (p \wedge q) \wedge \neg(p \wedge q)]$$

$$[(p \vee q) \wedge (\neg p \vee \neg q) \vee F] \vee [F \vee (p \wedge q) \wedge (\neg p \wedge \neg q)]$$

$$[(p \wedge \neg p) \vee (p \wedge \neg q) \vee (q \wedge \neg p) \vee (q \wedge \neg q)] \vee [(p \wedge \neg p) \wedge (p \wedge \neg q) \wedge (q \wedge \neg q) \wedge (q \wedge \neg p)]$$

$$[(p \wedge \neg p) \vee (p \wedge \neg q) \vee (q \wedge \neg p) \vee (q \wedge \neg q)] \vee [F \wedge (p \wedge \neg q) \wedge (q \wedge \neg p)]$$

$$(p \wedge \neg p) \vee (p \wedge \neg q) \vee (q \wedge \neg p) \vee (q \wedge \neg q) \vee F =$$

$$(p \wedge \neg p) \vee (p \wedge \neg q) \vee (q \wedge \neg p) \vee (q \wedge \neg q)$$

PDNF $\longleftrightarrow$ PCNF

$\downarrow$ $\qquad\qquad$ $\downarrow$

SOP $\qquad\qquad$ POS

$(P \wedge q) \vee (P \wedge q) \longleftrightarrow (P \vee q) \wedge (P \vee q)$

$P \rightarrow q \equiv \sim P \vee q$

$P \leftrightarrow q \equiv (P \rightarrow q) \vee \wedge (q \rightarrow P)$

$\sim(P \vee q) = \sim P \wedge \sim q$

$\sim (P \wedge q) \equiv \sim P \vee \sim q$

DATE : / /

PAGE NO. :

Q. Obtain PDNF of $P \to ((P \to q) \land (\sim q \lor \sim P))$ without truth table.

Ans - $P \to ((P \to q) \land (\sim q \lor \sim P))$

$\because \quad P \to q = \sim P \lor q$

$\equiv \sim P \lor ((\sim P \lor q) \land (\sim q \lor \sim P))$

$\cancel{\sim P \lor ((\sim P \land \sim q) \lor (q \land \sim q))}$

$\equiv \sim P \lor ((\sim P \lor q) \land (q \land P))$ $\quad \{\because \sim q \lor \sim P = q \land P\}$

$\sim P \lor ((\sim P \land q \land P) \lor (q \land q \land P))$ $\quad \left\{ \begin{array}{l} P \lor F = P \\ P \land \sim P \equiv F \end{array} \right\}$

$\cancel{\sim P \lor (q \land \sim q) \lor ((\sim P \land q \land P) \lor (q \land q \land P))}$ $\quad \left\{ \begin{array}{l} P \land T = P \\ P \lor \sim P = T \end{array} \right\}$

$\equiv \sim P \lor ((\sim P \land P \land q) \lor (q \land q) \land P)$

$\equiv \sim P \lor ((F \land q) \lor (q \land P))$

$\equiv \sim P \lor (F \lor q \land P)$

$\equiv \sim P \lor (q \land P)$

$\cancel{(\sim P \lor (q \land \sim q)) \lor (q \land P)}$

$\equiv \sim P \land (q \lor \sim q) \lor (q \land P)$

$\equiv (\sim P \land q) \lor (\sim P \land \sim q) \lor (q \land P)$

Q. The truth table ~~for~~ this statement s is given in the table shown below determine its principle (∧)conjective normal form (PCNF).

| p | q | r | s |
|---|---|---|---|
| T | T | T | F |
| T | T | F | F |
| T | F | T | T |
| F | T | T | T |
| T | F | F | F |
| F | T | F | T |
| F | F | T | F |
| F | F | F | T |

Ans - $(\sim P \lor \sim q \lor \sim r) \land (\sim p \lor \sim q \lor r)$ $\cancel{\land (\sim p \lor q \lor \sim r)} \cancel{\land (P \lor \sim q \lor \sim r)}$
$\land (\sim P \lor q \lor r)$ $\cancel{\land (P \lor \sim q \lor r)} \land (P \lor q \lor \sim r)$ $\cancel{\land (P \lor q \lor r)}$

DATE: / /

PAGE NO.:

Q Obtain PCNF of statements given by $(\neg p \rightarrow r) \wedge (q \leftrightarrow p)$ without using TT.

Ans-

$(\because\ p \rightarrow q = \neg p \vee q,\ q \leftrightarrow p = (p \rightarrow q) \wedge (q \rightarrow p))$

$\equiv (\neg(\neg p) \vee r) \wedge ((p \rightarrow q) \wedge (q \rightarrow p))$

$\equiv (\neg(\neg p) \vee r) \wedge ((\neg p \vee q) \wedge (\neg q \vee p))$

$(p \vee r) \wedge ((\neg p \vee q) \wedge (\neg q \vee p))$

$(p \vee r) \wedge (\neg p \vee q) \wedge (\neg q \vee p)$

$(p \vee r \vee (q \wedge \neg q)) \wedge (\neg p \vee q \vee (r \wedge \neg r)) \wedge (\neg q \vee p \vee (r \wedge \neg r))$

$((p \vee r \vee q) \wedge (p \vee r \vee \neg q)) \wedge ((\neg p \vee q \vee r) \wedge (\neg p \vee q \vee \neg r))$
$\wedge ((\neg q \vee p \vee r) \wedge (\neg q \vee p \vee \neg r))$

$(p \vee r \vee q) \wedge (p \vee \neg q \vee r) \wedge (\neg p \vee q \vee r) \wedge (\neg p \vee q \vee \neg r) \wedge (\neg q \vee p \vee r)$
$\wedge (p \vee \neg q \vee \neg r)$

Finite State Machine (FSM) ⇒

A device that receive a set of input signals and produces the corresponding set of output signal is termed as information processing machine.

A machine with a finite no. of states is called a finite state machine (FSM) while a machine has infinite no. of states is called as Infinite State Machine.

A finite state machine is an abstract model of machine with a premitive internal memory. A FSM M is specified by

DATE : / /
PAGE NO. :

(i) A finite set of input symbol $I = \{i_1, i_2, ---\}$
(ii) " " internal states $S = \{S_0, S_1, S_2 ---\}$
(iii) " " output symbol $O = \{O_0, O_1, O_2 ---\}$
(iv) An initial state (transition) function $f: S \times I \to S$
(v) An initial state $S_0$ in $S$
(vi) An output function $g: S \times I \to O$
A finite state machine $M = M(I, S, O, S_0, f, g)$

Eg Q. $I = \{a, b\}$, $S = \{S_0, S_1, S_2\}$, $O = \{X, Y, Z\}$
initial state $S_0$, transition function $f: S \times I \to S$
defined as $f(S_0, a) = S_1$, $f(S_1, a) = S_2$,
$f(S_2, a) = S_0$, $f(S_0, b) = S_2$, $f(S_1, b) = S_1$,
$f(S_2, b) = S_1$ & output function $g: S \times I \to O$
defined as $g(S_0, a) = x$, $g(S_1, a) = x$, $g(S_2, a) = Y$
$g(S_0, b) = Y$, $g(S_1, b) = Z$, $g(S_2, b) = x$.

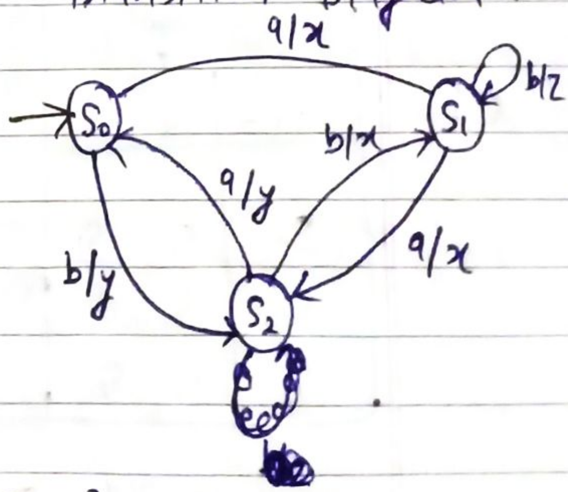
- Now draw the transition table & state diagram for the given FSM.

Ans –

Transition Table :–

| | f | | g | |
|---|---|---|---|---|
| S \ I | a | b | a | b |
| $S_0$ | $S_1$ | $S_2$ | x | y |
| $S_1$ | $S_2$ | $S_1$ | x | z |
| $S_2$ | $S_0$ | $S_1$ | y | x |

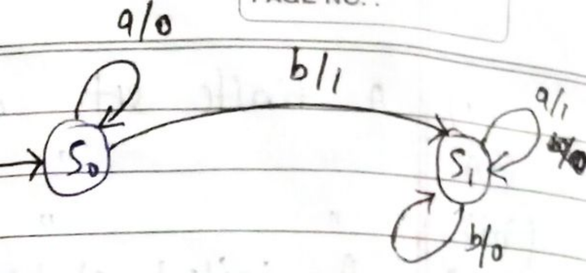
Transition Diagram :–




Q. $I = \{a, b\}$ $S = \{S_0, S_1\}$, $O = \{0, 1\}$
$f(S_0, a) = S_0$, $f(S_1, a) = S_1$, $f(S_0, b) = S_1$, $f(S_1, b) = S_1$
$g(S_0, a) = 0$, $g(S_1, a) = 1$ $g(S_0, b) = 1$, $g(S_1, b$

Ans –

| | f | | g | |
|---|---|---|---|---|
| S\I | a | b | a | b |
| $S_0$ | $S_0$ | $S_1$ | 0 | 1 |
| $S_1$ | $S_1$ | $S_1$ | 1 | 0 |




Input and output String ⟹

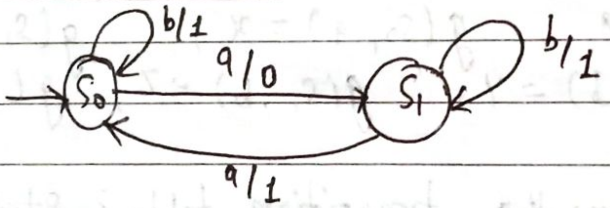
Q. find the output string if input string is bbaa

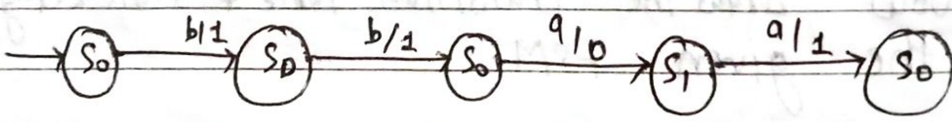
$I = \{a, b\}$ $S = \{S_0, S_1\}$, $O = \{0, 1\}$

$M \in (I, S, O, S_0, f, g)$

| | f | | g | |
|---|---|---|---|---|
| S\T | a | b | a | b |
| $S_0$ | $S_1$ | $S_0$ | 0 | 1 |
| $S_1$ | $S_0$ | $S_1$ | 1 | 0 |

Ans –



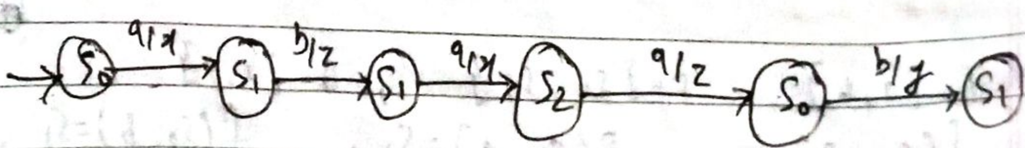




1 1 0 1

Q.

| | f | | g | |
|---|---|---|---|---|
| S\I | a | b | a | b |
| $S_0$ | $S_1$ | $S_1$ | x | y |
| $S_1$ | $S_2$ | $S_1$ | x | z |
| $S_2$ | $S_0$ | $S_1$ | z | y |

a b a a b

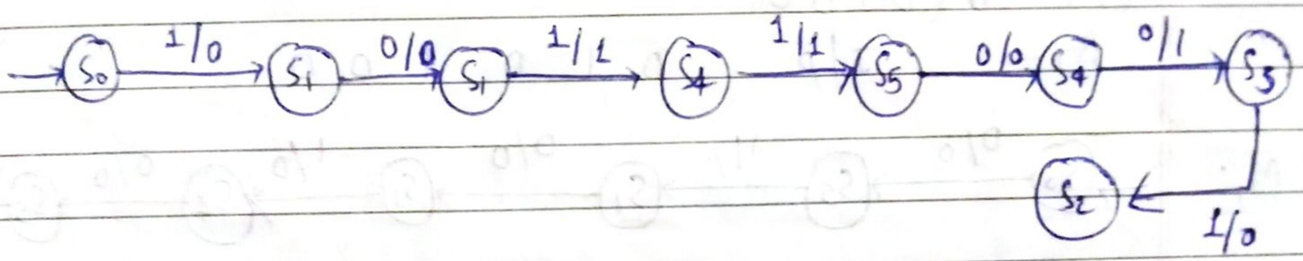



a b a a b = x z x z y Ans

| | f | | g | |
|---|---|---|---|---|
| | a | b | a | b |
| $S_0$ | $S_1$ | $S_3$ | 0 | 1 |
| $S_1$ | $S_4$ | $S_1$ | 1 | 0 |
| $S_2$ | $S_0$ | $S_1$ | 1 | 1 |
| $S_3$ | $S_2$ | $S_3$ | 0 | 1 |
| $S_4$ | $S_5$ | $S_3$ | 1 | 1 |
| $S_5$ | $S_1$ | $S_4$ | 0 | 0 |

1 0 11 0 0 1

aba




0 0 1 1 0 1 0 Ans

Equivalence Machines $\Longrightarrow$ Two machine $M_1$ & $M_2$ are called equivalent if starting from there ~~des~~ respective initial state will produce the same output. Although there internal structure is different. State $S_i$ and $S_j$ are said to be zero equivalent if they have same output. And one equivalent if they have same output and if for every input symbol there successor are zero equivalent.

Q. Consider two FSM whose transition table are as follow :-

$M_1$

| S\I | f | | g | |
|---|---|---|---|---|
| | 0 | 1 | 0 | 1 |
| $S_0$ | $S_5$ | $S_3$ | 0 | 1 |
| $S_1$ | $S_1$ | $S_4$ | 0 | 0 |
| $S_2$ | $S_1$ | $S_3$ | 0 | 0 |
| $S_3$ | $S_1$ | $S_2$ | 0 | 0 |
| $S_4$ | $S_5$ | $S_2$ | 0 | 1 |
| $S_5$ | $S_4$ | $S_1$ | 0 | 1 |

$M_2$

| s\I | f |  | g |  |
|---|---|---|---|---|
|  | 0 | 1 | 0 | 1 |
| $s_0'$ | $s_3'$ | $s_2'$ | 0 | 1 |
| $s_1'$ | $s_1'$ | $s_0'$ | 0 | 0 |
| $s_2'$ | $s_1'$ | $s_2'$ | 0 | 0 |
| $s_3'$ | $s_0'$ | $s_1'$ | 0 | 1 |

$M(I, S, O, s_0, f, g)$

(i) 010100

Ans - I/P 0 1 0 1 0 0

$M_1$: $\to S_0 \xrightarrow{0/0} S_5 \xrightarrow{1/1} S_1 \xrightarrow{0/0} S_1 \xrightarrow{1/0} S_4 \xrightarrow{0/0} S_5 \xrightarrow{0/0} S_4$

= 010000

$M_2$: $\to S_0' \xrightarrow{0/0} S_3' \xrightarrow{1/01} S_1' \xrightarrow{0/0} S_1' \xrightarrow{1/0} S_0' \xrightarrow{0/0} S_3' \xrightarrow{0/0} S_0'$

010000

So the machine $M_1$ & $M_2$ are equivalent.

(ii) 11010

$M_1$: $\to S_0 \xrightarrow{1/1} S_3 \xrightarrow{1/0} S_2 \xrightarrow{0/0} S_1 \xrightarrow{1/0} S_4 \xrightarrow{0/0} S_5$

10000

$M_2$: $\to S_0' \xrightarrow{1/1} S_2' \xrightarrow{1/0} S_2' \xrightarrow{0/0} S_1' \xrightarrow{1/0} S_0' \xrightarrow{0/0} S_3'$

10000

So the machine $M_1$ & $M_2$ are equivalent.

Q. Minimise the FSM given by the following state table:

| State | Input 0 | Input 1 | Output |
|---|---|---|---|
| A | D | B | 1 |
| B | E | B | 0 |
| C | D | A | 1 |
| D | C | D | 0 |
| E | B | A | 1 |

A -

$\Pi_0 = \{ \{A, C, E\}, \{B, D\} \}$

$\Pi_1 = \{ \{A\}, \{C, E\}, \{B, D\} \}$

$\Pi_2 = \{ \{A\}, \{C, E\}, \{B, D\} \}$

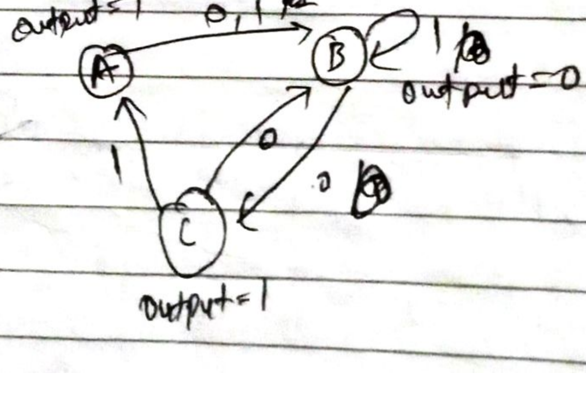
$\Pi_1 = \Pi_2 = ---- = \Pi_k$

A | D B
C | D A → not same set
same

A | D B
E | B A → not same set
same set

C | D A
E | B A
same set same

| State | Input 0 | Input 1 | Output |
|---|---|---|---|
| A | B | B | 1 |
| B | C | B | 0 |
| C | B | A | 1 |



Hasse's Diagram :-

Lattices:- A lattices is a partial poset $(A, \leq)$ in which every pair of element $(a,b) \in A$ has a greatest lower bound and least upper bound

LUB of $(a,b) \in A$ is denoted by $a \vee b$ and $\xrightarrow{}$ join

GLB of $(a,b) \in A$ is denoted by $a \wedge b$. $\xrightarrow{}$ meet

Some properties of lattices ⇒

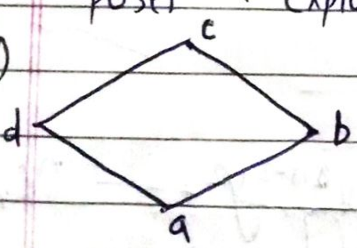
(i) $a \leq a \vee b$ and $b \leq a \vee b \Rightarrow a \vee b$ is an upper bound of $a$ & $b$.

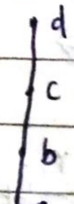
(ii) If $a \leq c$ and $b \leq c$ then $a \vee b \leq c \Rightarrow a \vee b$ is LUB of $a$ & $b$.

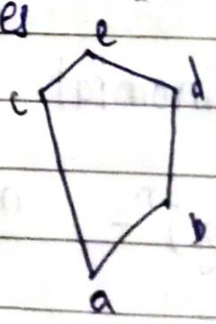
(iii) $a \wedge b \leq a$ and $a \wedge b \leq b \Rightarrow a \wedge b$ is an lower bound of $a$ & $b$.

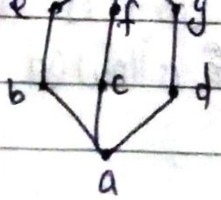
(iv) If $c \leq a$ & $c \leq b$ then $c \leq a \wedge b \Rightarrow a \wedge b$ is GLB of $a$ & $b$.

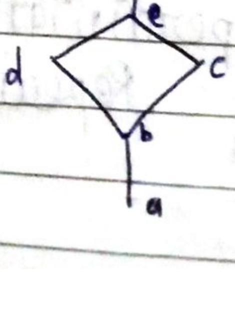
Q. The following are the hasse's diagram of some poset. Explain which are lattices

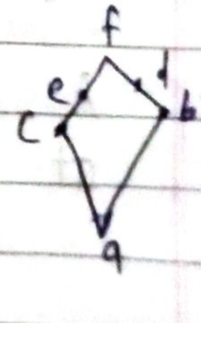
(i)



(ii)



(iii)



(iv)


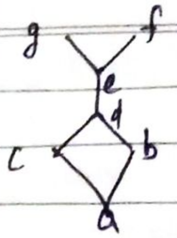

(v)



(vi)


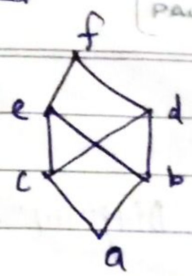
$\vee \longrightarrow$ disjunction $\longrightarrow$ sum $\longrightarrow$ join

$\wedge \longrightarrow$ conjunction $\longrightarrow$ product $\longrightarrow$ meet

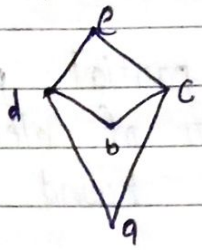
(vii)




(viii)




(ix)




Ans – (i) ✓ (ii) ✓ (iii) ✓ (iv) ✓ (v) ✓

(vi) ✓ (vii) ✗ (g & f are LUB)

(viii)

(ix) ✗ c, d have a & b LB

## Properties of Lattice ⇒

## Combinatorics :–

(i) Permutation ⇒ $\quad {}^nP_r = \dfrac{\lfloor n}{\lfloor n-r}$

(ii) Combination ⇒ $\quad {}^nC_r = \dfrac{\lfloor n}{\lfloor r \ \lfloor n-r}$

(iii) Binomial Theorem ⇒

$$(x+y)^n = {}^nC_0 x^n y^0 + {}^nC_1 x^{n-1} y^1 + {}^nC_2 x^{n-2} y^2 + \cdots {}^nC_n x^0 y^n = \boxed{\sum_{i=0}^{n} C(n,i)\, x^{n-i} y^i}$$

$\longrightarrow$ A no. of permutation of $n$ objects taken $r$ at a time. Repeatation is allowed is $n^r$

→ If Repeatation is ~~rep~~ not allowed then total no. of permutation :- $\frac{n!}{n_1!\, n_2! \cdots n_k!}$ where $n_1+n_2+n_3 \cdots + n_k = n$

→ The no. of circular permutation $\lfloor n-1$.

Q. A no consisting 4 digit is to be formed taken digit 1 to 9. How many such no. are possible?

Ans - $^9P_4 = \frac{\lfloor 9}{\lfloor 5} = 9 \times 8 \times 7 \times 6 = 72 \times 42$
$= 3024$

Q. If 4 digit no. is formed with digit 1 to 9 How many such no. can be fomed is repeated is allowed.?

Ans - $(9)^4 = 729 \times 9 = 6561$

Q. Find the no. of distincts permulation of the letter of the word ENGINEERING

Ans - $\frac{\lfloor 11}{\lfloor 3 \lfloor 3 \lfloor 2 \lfloor 2} = \frac{11 \times 10 \times \cancel{9}^{3} \times 8 \times 7 \times \cancel{6}^{2} \times 5}{\cancel{3} \times \cancel{2} \times 1 \times \cancel{3} \times \cancel{2} \times 1 \times \cancel{2} \times 2}$

$= 330 \times 7 \times 120 = 2310 \times 120$
$= 277200$

Q. $^{56}P_{r+6} : {}^{54}P_{r+3} = 30800 : 1$

$= \frac{\lfloor 56}{\lfloor 50-r} \times \frac{\lfloor 51-r}{\lfloor 54} = 30800$

$56 \times 55 \times (51-r) = 30800$

$3080 \times 51 - 3080r = 30800$ ⇒

$$154380 - 30800 = 3080r$$

$$\frac{123580}{3080} = r$$

$$\boxed{r = 41}$$

~~Q. How many permutation~~

Q. Prove that

$${}^{n}P_{r} = {}^{n-1}P_{r} + r\,{}^{n-1}P_{r-1}$$

Ans –

$$\frac{\lfloor n}{\lfloor n-r} = \frac{\lfloor n-1}{\lfloor n-r-1} + r\frac{\lfloor n-1}{\lfloor n-r}$$

$$\frac{n\,\cancel{\lfloor n-1}}{(n-r)\,\cancel{\lfloor n-r-1}} = \frac{\cancel{\lfloor n-1}}{\cancel{\lfloor n-r-1}}\left(1 + \frac{r}{n-r}\right)$$

$$\frac{n}{n-r} = 1 + \frac{r}{n-r}$$

$$\frac{n-r}{n-r} = 1 \quad \Rightarrow \quad \boxed{1 = 1}$$

$\boxed{\text{or}}$

$$\frac{\lfloor n}{\lfloor n-r} = \frac{\lfloor n-1}{\lfloor n-r-1}\left(\frac{n-r+r}{n-r}\right)$$

$$\frac{\lfloor n}{\lfloor n-r} = \frac{n\,\lfloor n-1}{(n-r)\,\lfloor n-r-1}$$

$$\frac{\lfloor n}{\lfloor n-r} = \frac{\lfloor n}{\lfloor n-r}$$

Q. Find the total no. of diagonal in Hexagone. If each diagonal combinding two vertex at the same time.

Ans – ${}^{6}C_{2} - 6 = \frac{3\cancel{6} \times 5}{\cancel{2}} - 6$

$15 - 6 = 9$ Ans

Q. How many combination are made with the letter of CONSTITUTION and find the no. of permutation & also find
(i) Two O come together
(ii) Vowel occur together
(iii) Consonant & vowel common alternative
(iv) Two O dont come together.
(v) letter N occur both at the begining & end

Ans — All possible $= \dfrac{\lfloor 12}{\lfloor 3 \lfloor 3 \lfloor 2 \lfloor 2 \lfloor 1} = 9979200$

(i) then $\dfrac{\lfloor 11}{\lfloor 3 \lfloor 2 \lfloor 2} = 1663200$

(ii) $\dfrac{\lfloor 8}{\lfloor 2 \lfloor 3} = 3360$ Also vowel $\dfrac{\lfloor 5}{\lfloor 2 \lfloor 2} = 30$

$P = 3360 \times 30 = 100800$

(iii) Vowel comes $2^{th}$ $4^{th}$ $6^{th}$ $8^{th}$ $10^{th}$

$\dfrac{\lfloor 5}{\lfloor 2 \lfloor 2} = 30$

So seven consonant $\dfrac{\lfloor 7}{\lfloor 3 \lfloor 2}$

$30 \times \dfrac{\lfloor 7}{\lfloor 3 \lfloor 2} = 12600$

(iv) $9979200 - 1663200 = 8316000$

(v) If N at begin & end fix then we have 10 lette

$\dfrac{\lfloor 10}{\lfloor 3 \lfloor 2 \lfloor 2} = 151200$

Q. Prove that

$$^{n}C_r = {}^{n-1}C_r + {}^{n-1}C_{r-1} \quad (\text{Pascal's identity})$$

<u>RHS</u>

$$\frac{\underline{|n-1}}{\underline{|r}\ \underline{|n-r-1}} + \frac{\underline{|n-1}}{\underline{|r-1}\ \underline{|n-r}}$$

$$\frac{\underline{|n-1}}{\underline{|r}\ \underline{|n-r-1}} + \frac{(n-1)!}{(r-1)!\,(n-r)\,\underline{|n-r-1}}$$

$$\frac{(n-1)!}{(n-r-1)!}\left(\frac{1}{r!} + \frac{1}{(n-r)\,(r-1)!}\right)$$

$$\frac{(n-1)!}{(n-r-1)!}\left(\frac{(n-r)\,(r-1)! + r!}{r!\,(n-r)\,(r-1)!}\right)$$

$$\frac{(n-1)!}{(n-r-1)!}\left(\frac{(n-r)\,(r-1)! + r\,(r-1)!}{r!\,(n-r)\,(r-1)!}\right)$$

$$\frac{(n-1)!}{(n-r-1)!}\left(\frac{n-r+r}{r!\,(n-r)}\right) = \frac{n\,(n-1)!}{r!\,(n-r)\,(n-r-1)!}$$

$$= \frac{n!}{r!\,(n-r)!} = {}^{n}C_r$$

<u>proved</u>

Q. <u>Recurrance Relation</u> ⇒

A relation that express $a_n$ in terms of one or more of the previous terms i.e. $a_0, a_1, \ldots a_{n-1}$ is called recurrance relation where $n$ is non-negative integer is called a <u>recurrance relation</u> for the sequence $\{a_n\}$.

For ex – Fibonacci sequence. (1, 1, 2, 3, 5, 8 - - -) can be defined by recurrance relation

$$a_n = a_{n-1} + a_{n-2}$$

with initial conditions $a_0 = 1$, $a_1 = 1$

Order of the Recurrance Relation :—

The difference of the greatest and smallest subscript appearing in recurrance relation is called its order.
For example, find the order of recurrance relation if $a_n = a_{n-1} - a_{n-2}$ then $O = n-(n-2)$
$= 2$

Solution to linear homogenous recurrance relation with constant coefficient ⇒

A linear homogenous recurrance relation of order k with constant coefficients is of form.

$$C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_n a_{n-k} \quad \text{— (I)}$$

Let the solution of eq$^n$-(I) be of the form $\boxed{a_n = r^n}$ where $r$ is a constant that must satisfy the given equation -(I)

$$C_0 r^k + C_1 r^{k-1} + \cdots + C_k r^{k-k} = 0 \quad \text{— (II)}$$

This eq$^n$ (II) is called characteristic eq$^n$ for relation first.
Let eq$^n$ -(II) is of $k^{th}$ degree so it has k roots. These roots are named as $r_1, r_2, r_3 \ldots$ $r_k$ roots. Now, the solution of relation (I) is dependent of the nature of these roots.

(1) When the roots are real & distinct

$$a_n = C_1 r_1^n + C_2 r_2^n + C_3 r_3^n + \cdots C_k r_k^n$$

(2) Real and 2 roots are equal
$r_1 = r_2 = r$ and $r_3, r_4$ are different

PAGE NO. :

$$a_n = (c_1 + c_2 n)\, r^n + c_3 r_3^n + \cdots$$

③ When the roots are imaginary and different $\alpha \pm i\beta$

$\theta = \tan^{-1}\frac{\beta}{\alpha}$

$$a_n = (\alpha^2+\beta^2)^{n/2}\,(c_1 \cos n\theta + c_2 \sin n\theta)$$

④ Imaginary & Repeated roots

$$a_n = (\alpha^2+\beta^2)^{n/2}\,[(c_1 + c_2 n)\cos n\theta + (c_3 + c_4 n)\sin n\theta]$$

Q. Solve the recurrence relation

$a_n + a_{n-1} - 6a_{n-2} = 0$ where $n \geq 2$ and $a_0 = 1$ & $a_1 = 2$

Ans- The characteristic eq$^n$ for the given relation

$$r^2 + r - 6 = 0 \quad [\because a_n = r^n,\ n=2]$$

$$r = 2, -3 \quad (\text{real \& distinct})$$

Here order $(n-(n-2)) = 2$

$$r^2 + r - 6 = 0$$

$$r = 2, -3$$

The sol$^n$ is $a_n = c_1 2^n + c_2(-3)^n$

if $n = 0$

$$a_0 = c_1 2^0 + c_2(-3)^0$$

$$a_0 = c_1 + c_2$$

$$1 = c_1 + c_2 \quad \text{—— ②}$$

if $n=1$

$$a_1 = c_1 2 + c_2(-3)$$

$$2 = 2c_1 - 3c_2 \quad \text{—— ③}$$

$C_1 = 1 - C_2$

$2 = 2(1-C_2) - 3C_2$

$C_2 = 0 \quad , C_1 = 1 \quad$ then $\boxed{a_n = 2^n}$

By $eq^n$-① we get the desired solution.

Q. Solve $a_{n+2} = a_{n+1} + a_n \quad n \geq 0$ and $a_0 = 1, a_1 = 2$

Ans – order ~~(n+2-n) = 2~~

$\cancel{r^{n+2} = r^{n+1} + r^n}$

$\cancel{r^3 = r^2 + r}$

$\cancel{r^3 - r^2 - r = 0}$

The characteristic $eq^n$ is

$r^2 - r - 1 = 0 \qquad [\, a_{n+2} = r^{n+2} \text{ or } r, \; n \geq 0 \,]$

$r = \frac{1 \pm \sqrt{5}}{2}$ ( Real & distinct ) $[\, 0 = n+2-n = 2 \,]$

The solution is

$$a_n = C_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + C_2 \left(\frac{1-\sqrt{5}}{2}\right)^n \text{ — ①}$$

Given that $a_0 = 1$ & $a_1 = 2$ : so (1) gives

$a_0 = C_1 + C_2 = 1$ — ②

$a_1 = C_1\left(\frac{1+\sqrt{5}}{2}\right) + C_2\left(\frac{1-\sqrt{5}}{2}\right) \qquad [\because C_1 = 1 - C_2]$

$2 = (1-C_2)\left(\frac{1+\sqrt{5}}{2}\right) + C_2\left(\frac{1-\sqrt{5}}{2}\right)$

$2 = \frac{1+\sqrt{5}}{2} - C_2\left(\frac{1+\sqrt{5}}{2} - \frac{1+\sqrt{5}}{2}\right)$

$2 = \frac{1+\sqrt{5}}{2} - \sqrt{5}C_2 \Rightarrow \sqrt{5}C_2 = \frac{1+\sqrt{5}}{2} - 2$

$C_2 = \frac{1+\sqrt{5}-4}{2\sqrt{5}} = \frac{\sqrt{5}-3}{2\sqrt{5}}$

& $C_1 = 1 - \left(\frac{\sqrt{5}-3}{2\sqrt{5}}\right) = \frac{2\sqrt{5}-\sqrt{5}+3}{2\sqrt{5}} = \frac{\sqrt{5}+3}{2\sqrt{5}}$

DATE : / /

PAGE NO. :

$\therefore$ By Eq$^n$ - (1) The required sol$^n$ is

$$a_n = \frac{3+\sqrt{5}}{2\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{\sqrt{5}-3}{2\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n \quad \text{Ans}$$

Solution of linear Non-Homogeneous Recurrence Relation with constant coefficient →

$$C_0 a_n + C_1 a_{n-1} + \cdots + C_k a_{n-k} = F(n) \quad \text{—— (1)}$$

where

$C_0, C_1, C_2 \cdots C_k$ are real no such that $C_0 \neq 0$ & $C_k \neq 0$ and $F(n)$ is a function of $n$ alone.

The sol$^n$ of eq$^n$ - (1)

$$a_n = a_n^{(h)} + a_n^{(P)} \quad \text{—— (2)}$$

where

$a_n^{(h)}$ is a sol$^n$ of following associated homogeneous recurrence relation.

$$C_0 a_n + C_1 a_{n-1} + \cdots C_k a_{n-k} = 0 \quad \text{—— (3)}$$

and $a_n^{(P)}$ is perticular sol$^n$ of (1).

There is no general method to find the perticular sol$^n$ of R.R. (1). It depends on the nature (form) of $F(n)$. Let in relation (1)

$$F(n) = (b_t n^t + b_{t-1} n^{t-1} + \cdots b_1 n + b_0)\, s^n$$

where $b_0, b_1, \cdots b_t$ & $s$ are real no. Now there are 2 possibilities for $s$, as given below.

Case-I Where $s$ is not a root of characteristics eq$^n$ of associated linear Homo. R.R.

DATE : / /

PAGE NO. :

In this case we assume that the form of particular sol$^n$ is

$$a_n^{(p)} = (P_t n^t + P_{t-1} n^{t-1} + \cdots P_1 n + P_0) s^n \quad (4)$$

Since $a_n^{(p)}$ is particular sol$^n$ of relation (1) so it most satisfied the relation (1). We substitute $a_n = a_n^{(p)}$ on LHS of (1) & then comparing the coefficient on both sides. We obtain the value of constants $P_0, P_1, \cdots P_t$ which on substitution in eq$^n$ – (4) give the desired particular solution.

Case-2

When $s$ is root of characteristic eq$^n$ of associated LHRR with multiplicity $m$.
In this case we assume that the form of a particular sol$^n$ is

$$a_n^{(p)} = n^m (P_t n^t + P_{t-1} n^{t-1} + \cdots P_1 n + P_0) s^n \quad (5)$$

where
$P_0, P_1, \cdots P_t$ are real no. determined by same procedure as given in case-I & $m$ is only ~~+~~ integer i.e. multiplicity of $n$.

Q. Solve the recurrence relation

$$a_n + 5a_{n-1} + 6a_{n-2} = 3n^2 - 2n + 1$$

Ans – The charateristic eq$^n$ is

$$r^2 + 5r + 6 = 0$$

$$\Rightarrow r = -2, -3$$

DATE : / /

PAGE NO. :

$\therefore \ a_n^{(h)} = C_1(-2)^n + C_2(-3)^n$ —— (1)

Now, here $F(n) = (3n^2 - 2n + 1)(1)^n$

The particular solⁿ of given relation is of the form

$$a_n^{(P)} = (P_2 n^2 + P_1 n + P_0) \text{ —— (2)}$$

{ Since here $t = 2$ (highest power of n) in $3n^2+2n+1$ & $s = 1$ which is not a root of characteristic eqⁿ so we apply case-I }

It must satisfy the given relation i.e.

$$(P_2 n^2 + P_1 n + P_0) + 5(P_2(n-1)^2 + P_1(n-1) + P_0) + 6(P_2(n-2)^2 + P_1(n-2) + P_0) = 3n^2 - 2n + 1$$

$$\Rightarrow n^2(P_2 + 5P_2 + 6P_2) + n(P_1 - 10P_2 + 5P_1 - 24P_2 + 6P_1) + (P_0 + 5P_2 - 5P_1 + 5P_0 + 24P_2 - 12P_1 + 6P_0) = 3n^2 - 2n + 1$$

On comparing the coefficient of equal power of n on both sides, we get

$$12P_2 = 3 \quad \Rightarrow \quad P_2 = \frac{1}{4} \text{ —— (3)}$$

and

$$P_1 - 10P_2 + 5P_1 - 24P_2 + 6P_1 = -2$$

$$12P_1 - 34P_2 = -2$$

$$12P_1 - 34\left(\frac{1}{4}\right) = -2 \quad \Rightarrow P_1 = \frac{13}{24} \text{ —— (4)}$$

and $P_0 + 5P_2 - 5P_1 + 5P_0 + 24P_2 - 12P_2 + 6P_0 = 1$

DATE : / /

PAGE NO. :

$$12P_0 - 17P_1 + 29P_2 = 1$$

$$12P_0 = 1 + 17\left(\frac{13}{24}\right) - 29\left(\frac{1}{4}\right)$$

$$P_0 = \frac{71}{288}$$

$\therefore$ Eq$^n$ – ② becomes

$$a_n^{(P)} = \frac{1}{4}n^2 + \frac{13}{24}n + \frac{71}{288}$$

Thus the complete sol$^n$ of the given relation is

$$a_n = a_n^{(h)} + a_n^{(P)}$$

$$\boxed{a_n = C_1(-2)^n + C_2(-3)^n + \frac{1}{4}n^2 + \frac{13}{24}n + \frac{71}{288}}$$ Ans

DATE : / /

PAGE NO. :

Q. Solve Recurrence Relⁿ

$a_n + a_{n-1} = 3n \cdot 2^n$

Ans - order $r = n - (n-1) = 1$

characteristics eqⁿ: -

$r + 1 = 0$

$r = -1$

$a_n^{(h)} = C(-1)^n$ —— (1)

Now Here $F(n) = 3n \cdot 2^n$

$a_n^{(p)} = (P_1 n + P_0) 2^n$ —— (2)

[On putting $t = 1$ & $s = 2$ in case - 1]

$a_n + a_{n-1} = 3n \cdot 2^n$

$(P_1 n + P_0) 2^n + (P_1(n-1) + P_0) 2^{n-1} = 3n \cdot 2^n$

$\Rightarrow (P_1 n + P_0) 2^n + \left(\frac{P_1 n}{2} - \frac{P_1}{2} + \frac{P_0}{2}\right) 2^n = 3n \cdot 2^n$

$$\left[\left(P_1 + \frac{P_1}{2}\right) n + \left(P_0 - \frac{P_1}{2} + \frac{P_0}{2}\right)\right] \cancel{2^n} = 3n \cdot \cancel{2^n}$$

$$\left(\frac{3P_1}{2} n + \frac{3P_0}{2} - \frac{P_1}{2}\right) = 3n + 0$$

$a_n = a_n^{(h)} + a_n^{(p)}$

compare the coefficient

$\frac{3P_1}{2} = 3$ , $\frac{3P_0}{2} - \frac{P_1}{2} = 0$

$\boxed{P_1 = 2}$ , $P_0 = \frac{P_1}{\cancel{2}} \times \frac{\cancel{2}}{3} \Rightarrow P_0 = \frac{P_1}{3}$

$\boxed{P_0 = \frac{2}{3}}$

By eqⁿ (2) we get

$a_n^{(p)} = \left(2n + \frac{2}{3}\right) 2^n$

DATE : / /

PAGE NO. :

$\therefore$ The complete solution

$$\boxed{a_n = C(-1)^n + \left(2n + \frac{2}{3}\right) 2^n}$$

Q. $a_n - 2a_{n-1} = 3(2)^n$

Ay- order $= n-(n-1) = 1$

$t=0,\ s=2$

$r^1 - 2r^0 = 0 \Rightarrow r = 2$

$a_n^{(h)} = C(2)^n$ —— ①

Here $F(n) = 3(2)^n$ [$\because P_t n^t 2^n = 3n^0 2^n$ i.e. $t=0$]

{i.e. here $t=0$ & $s=2$ which is root of char. eq$^n$ with multiplicity 1}

Particular sol$^n$ is

$a_n^{(P)} = n^1 P_0 2^n$ [$\because t=0, m=1, s=2$]

It must satisfy the given relation so

$nP_0 2^n - 2(n-1)P_0 2^{n-1} = 3 \cdot 2^n$

$(P_0 - P_0) n 2^n + P_0 2^n = 3 \cdot 2^n$

$P_0 2^n = 3 \cdot 2^n$

$P_0 = 3$

Eq$^n$ ② becomes $a_n^{(P)} = n \cdot 3 2^n$

Thus the complete sol$^n$ is

$a_n = a_n^{(h)} + a_n^{(P)}$

$$\boxed{a_n = C(2)^n + 3n2^n}$$

DATE : / /

PAGE NO. :

## Discrete Numeric function ⇒

The class of functions whose domain is the set of natural no. & whose range is set of real number, then these functions are called discrete numeric functions.

it can be written as.

$a = (a_0, a_1, a_2 -- a_r)$

where $a_0, a_1, a_2 -- a_r$ are values of function at $0, 1, 2 --- r$.

### Generating function:-

These are used to represent numeric functions in series form

let $a = (a_0, a_1, --- a_n)$ be a numeric function or be sequence of real no. Then the generating functions for this sequence can be written in series form as

$$A(x) = a_0 + a_1x + a_2x^2 + --- a_rx^r = \sum_{r=0}^{\infty} a_rx^r$$

The expression $A(x)$ is called a formal power series and the formal power series is called a generating function.

Q. Find G.F. for sequence 1,1,1,1 - - -

Sol^n^ $G(x) = 1 + a_1x + a_2x^2 ---$

$= 1 + x + x^2 + --- \quad \{\because a_0 = a_1 - --a_r = 1\}$

$= (1-x)^{-1}$

$= \dfrac{1}{1-x}$ [for $|x| < 1$]

[The sum of G.P. series with common ratio = x]

DATE: / /
PAGE NO.:

**Properties of G.F. →**

Let $A(x) = \sum_{k=0}^{\infty} a_k x^k$ & $B(x) = \sum_{k=0}^{\infty} b_k x^k$ the generating fun. of numeric function

$a = (a_0, a_1, a_2 \dots a_k \dots)$ and
$b = (b_0, b_1, b_2 \dots b_k \dots)$ respectively.

Then

(i) $A(x) = B(x)$ if $a_n = b_n$ for each $n \geq 0$

(ii) $cA(x) = \sum_{k=0}^{\infty} (ca_k) x^k$, where $c$ is any scalar no.

(iii) $A(x) = B(x) = \sum_{k=0}^{\infty} (a_k + b_k) x^k$

(iv) $A(x)B(x) = \sum_{k=0}^{\infty} \left( \sum_{j=0}^{\infty} a_j b_{k-j} \right) x^k$

Q. Express $F(x) = \frac{1}{(1-x)^2}$ in summation form

Ans- Let $f(x) = \frac{1}{1-x}$ & $g(x) = \frac{1}{1-x}$ then

$f(x) = \frac{1}{1-x} = 1 + x + x^2 \dots = \sum_{k=0}^{\infty} x^k$ — (1)

& $g(x) = \frac{1}{1-x} = 1 + x + x^2 \dots = \sum_{k=0}^{\infty} x^k$ — (2)

By (iv) property

$F(x) = \frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} \left( \sum_{j=0}^{\infty} a_j b_{k-j} \right) x^k$

Since (1) ⇒ $a_k = 1 \ \forall j$ & (2) ⇒ $b_k = 1 \ \forall k$

$\therefore \frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} \left( \sum_{j=0}^{k} 1 \right) x^k = \sum_{k=0}^{\infty} (k+1) x^k$

DATE : / /

PAGE NO. :

## Extended Binomial coefficient :—

Let $P$ be a real no. and $k$ be a non-negative integer then the extended Binomial coefficient ${}^{P}C_{x}$ is defined as

$$ {}^{P}C_{k} = \begin{cases} \dfrac{P(P-1)\cdots(P-k+1)}{k!} & , k>0 \\ 1 & , k=0 \end{cases} $$

$$ {}^{P}C_{k} = \frac{P(P-1)(P-2)\cdots(P-k+1)(P-k)}{\lfloor k \, \lfloor P-k} $$

if $k=0$ ${}^{P}C_{0} = \dfrac{\lfloor P}{\lfloor 0 \, \lfloor P-1} = 1$

If $P$ is $-ve$ then EBC can be described in terms of an ordinary B.C. as given below

$$ {}^{-n}C_{r} = \frac{-n(-n-1)\cdots(-n-r+1)(-n-r)!}{\lfloor r \, \lfloor -n-r} $$

$$ = (-1)^{r} \frac{n(n+1)\cdots(n+r-1)}{\lfloor r} $$

$$ = (-1)^{r} \frac{(n+r-1)(n+r-2)\cdots(n+1)n}{\lfloor r} $$

multipled & divide by $\left\{ \dfrac{(n-1)\cdots 3.2.1}{(n-1)\cdots 3.2.1} \right.$

$$ = (-1)^{r} \frac{\lfloor n+r-1}{\lfloor r \, \lfloor n-1} $$

$$ \boxed{ {}^{-n}C_{r} = (-1)^{r} \; {}^{n+r-1}C_{r} } $$

Q. Evaluate (i) ${}^{-2}C_{3}$ (ii) ${}^{3/2}C_{3}$

(i) ${}^{-2}C_{3} = (-1)^{3} \; {}^{2+3-1}C_{3} = (-1) \; {}^{4}C_{3} = -4$ Ans

(ii) $$ {}^{3/2}C_{3} = \frac{\frac{3}{2}\left(\frac{3}{2}-1\right)\left(\frac{3}{2}-2\right)\cdots\left(\frac{3}{2}-3+1\right) \cancel{\lfloor \frac{3}{2}-3}}{\lfloor 3 \, \cancel{\lfloor \frac{3}{2}-3}} $$

$$ = \frac{\frac{3}{2} \cdot \frac{1}{2} \cdot \left(-\frac{1}{2}\right)}{6} = \frac{-3}{8 \times 6} = -\frac{1}{16} \text{ Ans} $$

DATE : / /

PAGE NO. :

**Theorem:– Extended Binomial Theorem**

Let $x$ be real no. such that $|x|<1$ & also let $p$ be a real no. then

$$\boxed{(1+x)^p = \sum_{k=0}^{\infty} {}^{p}C_k\, x^k}$$

Q. Evaluate the G.F. for $(1+x)^{-n}$ & $(1-x)^{-n}$ where $n$ is +ve.

Ans– $(1+x)^{-n} = \sum_{k=0}^{\infty} {}^{-n}C_k\, x^k = \sum_{k=0}^{\infty} (-1)^k\, {}^{n+k-1}C_k\, x^k$

Now replacing $x$ by $-x$, we get

$$(1-x)^{-n} = \sum_{k=0}^{\infty} (-1)^k\, {}^{n+k-1}C_k\, (-x)^k$$

$$= \sum_{k=0}^{\infty} (-1)^k\, {}^{n+k-1}C_k\, (-1)^k (x)^k$$

$$= \sum_{k=0}^{\infty} {}^{n+k-1}C_k\, (x)^k$$

**Exponential Generating function:–**

$$\boxed{{}^{n}C_r = \frac{\lfloor n}{\lfloor r \lfloor n-r} = \frac{1}{\lfloor r}\, {}^{n}P_r}$$

Since, we have

$(1+x)^n = {}^{n}C_0 + {}^{n}C_1 x + {}^{n}C_2 x^2 + \cdots {}^{n}C_r x^r + \cdots {}^{n}C_n x^n$

so by method

$$(1+x)^n = {}^{n}P_0 \frac{x^0}{\lfloor 0} + {}^{n}P_1 \frac{x^1}{\lfloor 1} + {}^{n}P_2 \frac{x^2}{\lfloor 2} + \cdots {}^{n}P_r \frac{x^r}{\lfloor r} + \cdots {}^{n}C_n \frac{x^n}{\lfloor n}$$

Here it is clear that coefficient of $\frac{x^r}{\lfloor r}$

Gives the sequence ${}^{n}P_r$

Now let,

DATE : / /

PAGE NO. :

$$f(x) = a_0 + a_1 x + a_2 \frac{x^2}{\lfloor 2} + a_3 \frac{x^3}{\lfloor 3} + \cdots = \sum_{k=0}^{\infty} \frac{a_k x^k}{\lfloor k}$$

then $f(x)$ is called E.G.F for sequence $a_0, a_1, \ldots a_k$

Application of Generating function ⇒

| | $G(x)$ | | Coefficient of $x^k$ |
|---|---|---|---|
| (i) | $(1+x)^n = \sum_{k=0}^{n} {}^nC_k x^k$ | = | ${}^nC_k$ |
| (ii) | $(1+ax)^n = \sum_{k=0}^{n} a^k \, {}^nC_k x^k$ | = | $a^k \, {}^nC_k$ |
| (iii) | $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ | = | $1$ |
| (iv) | $\frac{1}{1-ax} = \sum_{k=0}^{\infty} a^k x^k$ | = | $a^k$ |
| (v) | $\frac{1}{(1-ax)^n} = \sum_{k=0}^{\infty} {}^{-n}C_k (-a^k) x^k$ | = | ${}^{-n}C_k (-a)^k = a^k \, {}^{n+k-1}C_k$ |
| (vi) | $\frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k$ | = | $(-1)^k$ |
| (vii) | $\frac{1}{(1+ax)^n} = \sum_{k=0}^{\infty} {}^{-n}C_k a^k x^k$ | = | $(-1)^k a^k \, {}^{n+k-1}C_k$ |
| (viii) | $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ | = | $\frac{1}{k!}$ |

DATE : / /

PAGE NO. :

Q. Solve the RR $a_n = 3a_{n-1}$, $n = 1, 2, 3 ---$ with $a_0 = 2$ by the method of generating function

A- $G(x) = \sum_{n=0}^{\infty} a_n x^n$ —①

$a_n = 3a_{n-1}$ —②

Now multiply eqⁿ-② by $x^n$ & taking summation over $n=1$ to $\infty$

$\sum_{n=1}^{\infty} a_n x^n = 3 \sum_{n=1}^{\infty} a_{n-1} x^n$

or $a_1 x + a_2 x^2 + -- = 3[a_0 x + a_1 x^2 + ---]$ —③

Since $G(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 ---$

$G(x) - a_0 = a_1 x + a_2 x^2 ----$

from eqⁿ -③

$G(x) - a_0 = 3x[a_0 + a_1 x + a_2 x^2 ---]$

$G(x) - a_0 = 3x\, G(x)$

$G(x)[1 - 3x] = a_0$

$G(x) = \frac{2 \times 1}{1 - 3x}$

$G(x) = 2 \sum_{n=0}^{\infty} (3x)^n = \sum_{n=0}^{\infty} (2 \cdot 3^n) x^n$ —④

On comparing ① & ④ we get required solⁿ

$\boxed{a_n = 2 \cdot 3^n}$

Q. Solve RR using GF $a_n = 3a_{n-1} - 2a_{n-2}$, $n \geq 2$ with $a_1 = 5$, $a_2 = 3$

A- $G(x) = \sum_{n=0}^{\infty} a_n x^n$ —①

or $G(x) = a_0 + a_1 x + a_2 x^2 ---$ —②

Given $a_n = 3a_{n-1} - 2a_{n-2}$ —③

multiply (3) by $x^n$ & taking summation over $n=3$ to $\infty$

$$\sum_{n=3}^{\infty} a_n x^n = 3\sum_{n=3}^{\infty} a_{n-1} x^n - 2\sum_{n=3}^{\infty} a_{n-2} x^n$$

or

$$a_3x^3 + a_4x^4 + \cdots = 3[a_2x^3 + a_3x^4 \cdots] - 2[a_1x^3 + a_2x^4 + \cdots]$$

or

$$G(x) - a_0 - a_1x - a_2x^2 = 3x[a_2x^2 + a_3x^3] - 2x^2(a_1x + a_2x^2 + \cdots$$

$$G(x) - a_0 - a_1x - a_2x^2 = 3x[G(x) - a_0 - a_1x] - 2(x^2)[G(x) - a_0] \quad \text{—— (4)}$$

From (3)

$$a_2 = 3a_1 - 2a_0 \Rightarrow 2a_0 = 3a_1 - a_2$$

$$a_0 = \frac{3\times5-3}{2} = 6$$

Eq$^n$ (4) becomes

$$G(x) - 6 - 5x - 3x^2 = 3x[G(x) - 6 - 5x] - 2x^2[G(x) - 6]$$

$$G(x)(1 - 3x + 2x^2) = -18x - 15x^2 + 12x^2 + 6 + 5x + 3x^2$$

$$G(x) = \frac{6-13x}{1-3x+2x^2} = \frac{6-13x}{(x-1)(2x-1)}$$

By partial fraction

$$\frac{1}{2x-1} - \frac{7}{x-1}$$

$$G(x) = \frac{-1}{1-2x} + \frac{7}{1-x}$$

$$G(x) = 7\sum_{n=0}^{\infty} x^n - \sum_{n=0}^{\infty} (2x)^n$$

$$= \sum_{n=0}^{\infty} (7 - 2^n) x^n \quad \text{—— (5)}$$

On comparing ① & ⑤ we get required solⁿ :-

$\boxed{a_n = 7 - 2^n}$

Q. Determine the coefficient of $x^8$ in $\frac{1}{(x-3)(x-2)^2}$

Solⁿ:- Since $\frac{1}{(x-3)(x-2)^2} = \frac{1}{x-3} - \frac{1}{x-2} - \frac{1}{(x-2)^2}$ By partial fraction

$(x-3)^{-1}$
$= (-3)^{-1}\left(1-\frac{x}{3}\right)^{-1} = \frac{-1}{3}\left(1-\frac{x}{3}\right)^{-1} + \frac{1}{2}\left(1-\frac{x}{2}\right)^{-1} - \frac{1}{4}\left(1-\frac{x}{2}\right)^{-2}$

so, coeff. of $x^8$ in $\frac{1}{(x-3)(x-2)^2}$

$= \text{coeff of } x^8 \text{ in}\left[-\frac{1}{3}\left(1-\frac{x}{3}\right)^{-1}\right] + \text{coeff of } x^8 \text{ in}\left[\frac{1}{2}\left(1-\frac{x}{2}\right)^{-1}\right]$

$+ \text{ coeff of } x^8 \text{ in}\left[-\frac{1}{4}\left(1-\frac{x}{2}\right)^{-2}\right]$

$= \left(\frac{-1}{3}\right)^{(1+8-1)}C_8\left(\frac{1}{3}\right)^8 + \left(\frac{1}{2}\right)^{1+8-1}C_8\left(\frac{1}{2}\right)^8 + \left(\frac{-1}{4}\right)^{2+8-1}C_8\left(\frac{1}{2}\right)^8$

$= -\frac{1}{3}\ {}^8C_8\left(\frac{1}{3}\right)^8 + \frac{1}{2}\ {}^8C_8\left(\frac{1}{2}\right)^8 + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\ {}^9C_8\left(\frac{1}{2}\right)^8$

$= -\left(\frac{1}{3}\right)^9 + \left(\frac{1}{2}\right)^9 - \frac{9}{2}\left(\frac{1}{2}\right)^9$

$= \frac{-1}{3^9} + \frac{1}{2^9}\left(1-\frac{9}{2}\right) = \frac{-1}{3^9} + \frac{1}{2^9}\left(\frac{-7}{2}\right)$

$= -\left[\frac{1}{3^9} + \frac{7}{2^{10}}\right]$ Ans

Q. In how many ways can 4 of the alphabets in BETTER be arranged?

Solⁿ- Here we apply exponential generating function.

The factor corresponding to B is $(1+x)$, the factor corr. to E is $\left(1+x+\frac{x^2}{\lfloor 2}\right)$, the factor corresponding to T is $\left(1+x+\frac{x^2}{\lfloor 2}\right)$ & the factor corr. to R is $(1+x)$

Thus EGF is

$$G(x) = (1+x)^2\left(1+x+\frac{x^2}{\lfloor 2}\right)^2$$

BETTER
B = 1 time i.e. $(1+x)$
E = 2 time i.e. $\left(1+x+\frac{x^2}{\lfloor 2}\right)$
T = 2 time i.e. $\left(1+x+\frac{x^2}{\lfloor 2}\right)$
R = 1 time i.e. $(1+x)$

Now the required no. of ways

= coeff. of $\frac{x^4}{\lfloor 4}$ in expansion of $G(x)$

= Coeff. of $\frac{x^4}{\lfloor 4}$ in $(1+x)^2\left(1+x+\frac{x^2}{\lfloor 2}\right)^2$

= coeff. of $\frac{x^4}{4!}$ in $(x^2+2x+1)\left(\frac{x^4}{4}+x^3+2x^2+2x+1\right)$

since coeff of $\frac{x^4}{\lfloor 4}$ in given term is

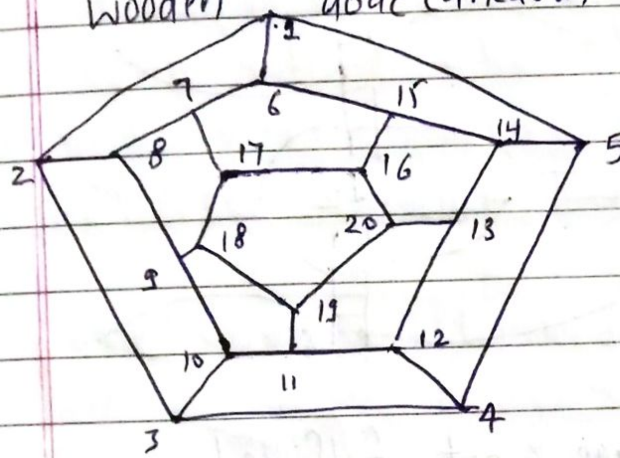
$\left(2+2+\frac{1}{4}\right) = \frac{17}{4}$ [$\because$ coeff of $x^4$ is $\frac{17}{4}$

coeff of $\frac{x^4}{\lfloor 4} = 4! \times \frac{17}{4} = 102$

$\therefore$ The required no. of ways 102 An

# GRAPH THEORY

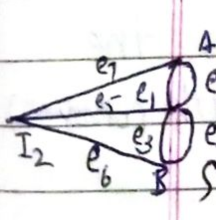
**Hamiltonian circuit :—** A puzzle game was invented in 1859 by Irish mathematician sir W.R. Hamilton (1805-1865). It consisted of a wooden dodecahedron having 12 faces & 20 corners




Each face was a regular pentagon with 3 edges meeting at each corner. The corner were marked with the names of 20 important cities like london, paris, delhi, newyork etc.

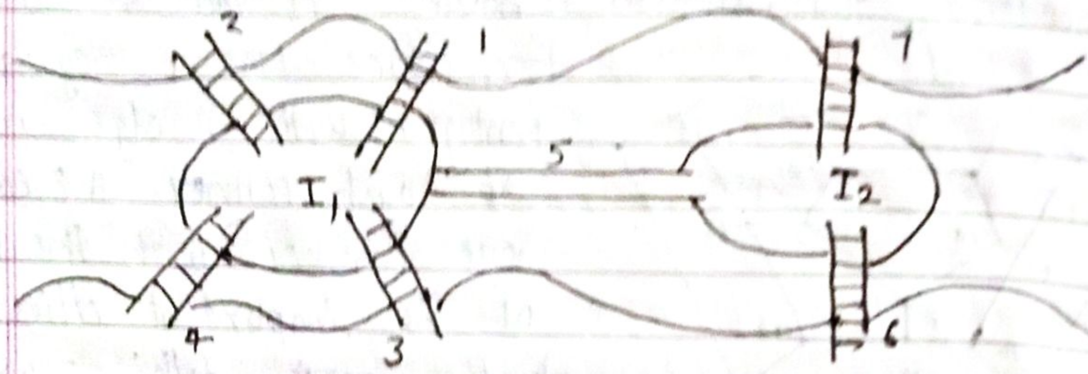
The object of puzzle was to start from any city & travel along the edges of the dodecahedran, visiting each of remaining 19 cities exactly once & return back to the starting city. Although the puzzle was easily solved but the necessary & sufficient condition for existence of such a circuit (called hamiltonian circuit) in an arbitrary graph is still the area for research.

**Eulerian Path ⇒** Here $A, B, I_1$ & $I_2$ are called vertices of graph, representing some points on banks A & B & that on islands $I_1$ & $I_2$ resp. The arcs on lines denoted by $e_1$ $e_2$ $e_3$ $e_4$ $e_5$ $e_6$ & $e_7$ are called edges of graph denoting seven bridges. The degree of vertex means the no. of edges incident on it




∴ the vertex $A, B, I_2$ were each of degree 3 & $I_1$ was of degree 5.

Then he defined a path called 'Eulerian path' such that each edge was traversed only once. He proved a theorem that an 'Eulerian circuit' (closed path) is possible only when each vertex is of even degree.




Since the condition was not fulfilled by the konigsberg problem so it could not be possible to have a Euler circuit in which each bridge be traversed only once & Thus the problem was settled down with conclusion that, it is not possible to have such a path as needed ie. "Euler circuit".

2 Famous unsolved problem in graph theory →

(i) The four color conjecture → In which the countries with common boundaries have different colors

(ii) Hamiltonian Circuit.

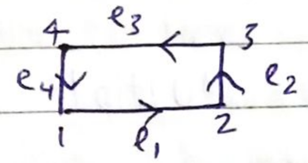
Graph $G = (V, E)$ consist two components

(i) The set of vertices $V$, also called points or nodes.
(ii) The set of edges $E$, also called lines or arcs connecting pair of vertices, each pair of vertices, connected by an edge, is called the end points or end vertices

**Directed Graph $G = (V, E)$ :-**

A directed graph $(V, E)$ consists of a set of vertices $V$ and a set of edges $E$. That are ordered pairs of elements of $V$.

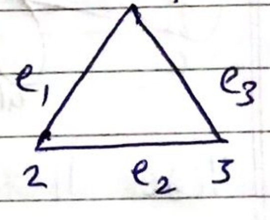
Eg - $G = (\{1, 2, 3, 4\}, \{e_1, e_2, e_3, e_4\})$




**Undirected Graph:-**

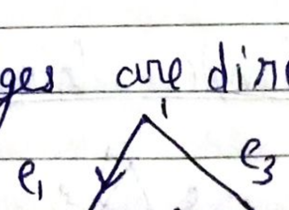
Each edge is associated with an unordered pair of vertices.

$G = (\{1, 2, 3\} \{e_1 e_2 e_3\})$



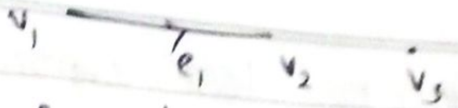

**Mixed Graph:-** If some edges are directed & some are undirected.

$G = (\{1, 2, 3\} \{e_1 e_2 e_3\})$



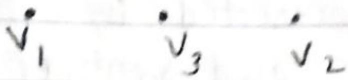

**Isolated vertex:-** A vertex $V$ which is not connected with any vertex of graph $G$ by an edge.

DATE: / /
PAGE NO:



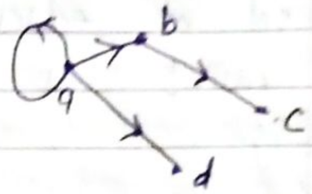

$V_3$ is isolated vertex

**Null Graph:-**
If all vertices of graph are isolated [i.e. set $E = \phi$] totally disconnected graph $G = (\{v_1, v_2, v_3\}, \phi)$

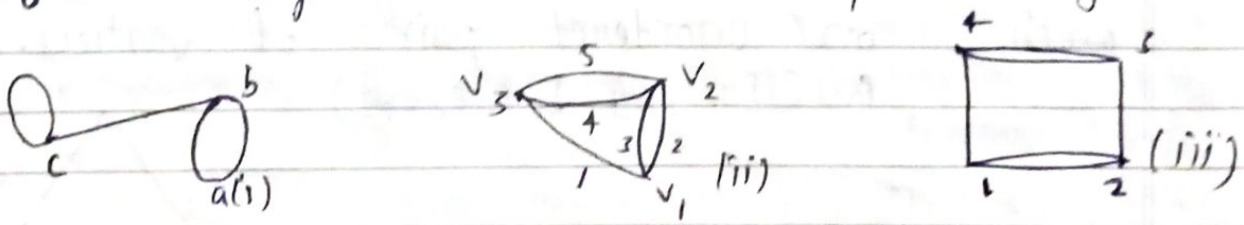



**Self loop:-** An edge having same vertex as both its end vertices i.e. (a,a)

$V = (a,b,c,d)$ & $E = \{(a,a)\ (a,b)\ (a,d)\ (b,c)\}$
The edge (a,a) → closed curve
↑
(self loop)

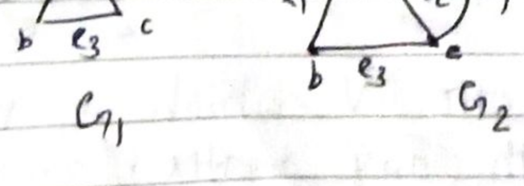



**Parallel Edge or Multiple Edges:-**
If a pair of vertices is joined by more than one edge then these edges are called parallel edges.




Here ∥ edge are (i) a,b (ii) 2,3 & 4,5 (iii) 1,2 & 3,4

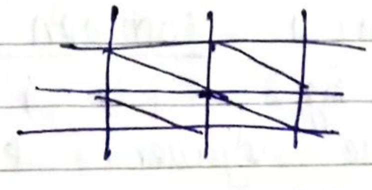
**Simple Graph:-** Neither self loop nor parallel edge.




$G_1 \to$ simple Graph
$G_2 \to$ multi Graph due to $e_2$ & $e_4$ parallel edges

**Finite & infinite Graph:-** In graph if set V & E are both finite then it is a finite graph otherwise an infinite graph.

eg -

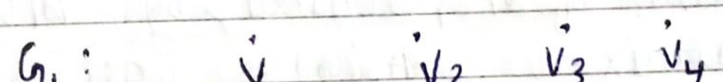


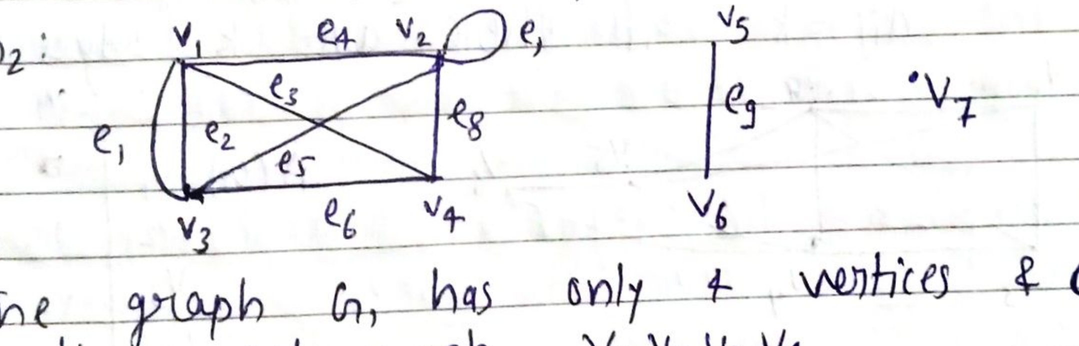
$G(V,E)$ where $V = \{v_1, v_2, v_3 \ldots\}$ and $E = \{e_1, e_2, e_3 \ldots\}$ then these are infinite Graphs.

Order of Graph $\longrightarrow$ The total no. of vertices in a graph.

Size of Graph $\longrightarrow$ The total no. of edges in a graph.

Q. Find no. of vertices, edges, loops, isolated vertex, paralled edges



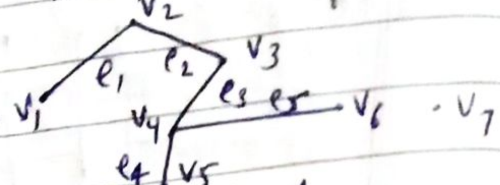

Sol$^n$- The graph $G_1$ has only 4 vertices & denoted a disconnected graph $v_1\ v_2\ v_3\ v_4$.

The graph $G_2 = \{(v_1\ v_2\ v_3\ v_4\ v_5\ v_6\ v_7\}\{e_1\ e_2\ e_3\ e_4\ e_5\ e_6\ e_7\ e_8\ e_9\})$

has (i) seven vertes
(ii) 9 edgs (iii) One self loop $e_7$
(iv) One isolated vertex $v_7$ (v) A pair of $\|^{rl}$ edges $e_1$ & $e_2$ with end vertex $v_1$ & $v_3$

Adjacent Edges & Adjacent vertices:-

2 non parallel edges are said to be adjacent if both are incident on a <u>common vertex.</u>




In fig $e_1$ & $e_2$ ; $e_2$ & $e_5$ ; $e_3, e_4, e_5$ are adjacent edges.

Two vertices connected by an edge are called adjacent vertices

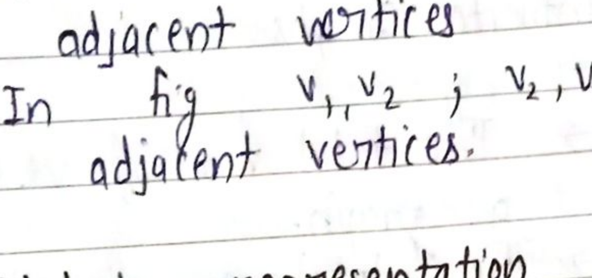
In fig $v_1, v_2$ ; $v_2, v_3$ ; $v_3, v_4$ ; $v_4, v_5$ ; $v_4, v_6$ are adjacent vertices.

Matrix representation of Graphs:-

(i) Adjacent Matrix:- Let G be a graph with n vertices then the adjanceny matrix of G, denoted by A(G) is symmetric matrix $A(G) = [a_{ij}]_{n \times n}$ where

$a_{ij} = k$ if there are k edges b/n $V_i$ & $V_j$

Eg -



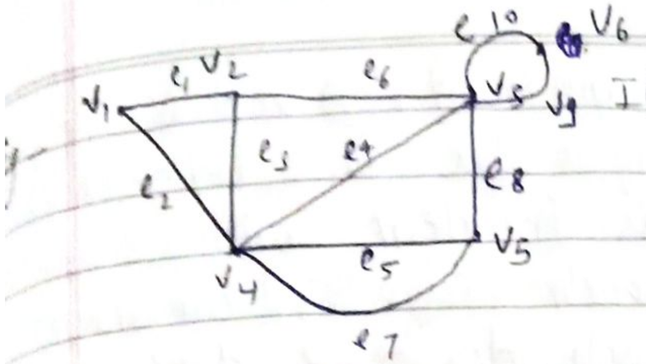

$$A(G) = \begin{array}{c|cccccc} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\ \hline v_1 & 0 & 1 & 1 & 0 & 1 & 0 \\ v_2 & 1 & 0 & 1 & 1 & 1 & 0 \\ v_3 & 1 & 1 & 0 & 1 & 0 & 0 \\ v_4 & 0 & 1 & 1 & 0 & 1 & 1 \\ v_5 & 1 & 1 & 0 & 1 & 0 & 1 \\ v_6 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}$$

(ii) Incidence Matrix:-

Let G be graph with n vertices, e edges and no self loops. Then the incidence matrix of G, denoted by $I(G) = (a_{ij})_{n \times e}$ is an $n \times e$ matrix where

$$a_{ij} = \begin{cases} 1 & \text{if } j \text{ edge } e_j \text{ is incident on } i\text{th vertex } V_i \\ 0 & \text{otherwise} \end{cases}$$

eg-

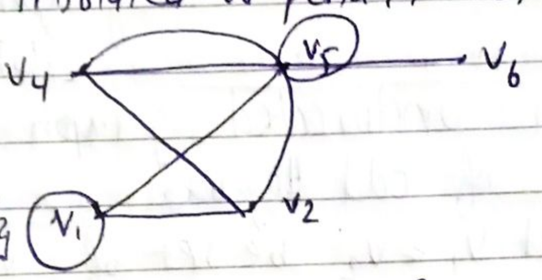



$$I(G) = \begin{array}{c|cccccccccc} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} \\ \hline v_1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ v_2 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ v_3 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ v_4 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ v_5 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ v_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}$$

<u>Degree of vertex</u> :- The no. of incident on a vertex v of graph is called the degree of vertex & is denoted by deg(v).

<u>Pendant vertex</u>:- A vertex of degree <u>one</u>.

eg- Find deg(v) & insolated or pendant vertex.




$deg(v_1) = 4$ { loop count 2 edges 1 for outdegree 1 for indegree }

$deg(v_2) = 3$  $deg(v_3) = 0$ { insolated vertex }
$deg(v_4) = 3$  , $deg(v_5) = 7$  , $deg(v_6 = 1)$ [Pendant vertex]

<u>Even-Odd vertices</u>:- In graph, if degree of vertex is an even integer then vertex is called <u>even vertex</u> and if degree of vertex is an odd integer then vertex is called <u>odd vertex</u>.

★ Hand Shaking Theorem ⇒ The sum of degrees of all vertices in graph is equal to twice the no. of edges in the graph

**Proof** :- Let $G=(V,E)$ be a graph with $n$ vertices i.e. $|V|=n$ & the no. of edges are $|E|=e$ (say)
Since each edge is incident with 2 vertices $(u,v)$ say so each edge is contributes a count of 1 to each deg $(v)$ & deg $(u)$
$\therefore$ e edges will contribute $2e$ degree (one for each end vertices of an edge) for all vertices.
Thus,

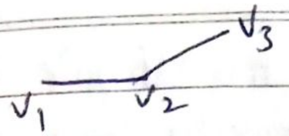
$$\boxed{\sum_{i=1}^{n} Deg(v_i) = 2e = 2|E|}$$

☆ Theorem :⟶ A no. of vertex of odd degree in graph G is always even.
OR
An undirected graph has even no. of vertices of odd degree.

Proof:— let $V_1$ & $V_2$ be set of vertices of even degree & the set of vertices of odd degree, respt. in undirected graph $G=(V,E)$ then

$$\boxed{2e = \sum_{v=V} deg(v) = \sum_{v\in V_1} deg(v) + \sum_{v\in V_2} deg(v)}; \quad e \in E$$

Since $deg(v)$ is even for $v \in V_1$ the first term in RHS of equality is even. Further more, the sum of two terms on right HS of equality is even. Since the sum is $2e$. Hence the second term in sum is also even.
Since all terms in this sum are odd there must be an even no. of such terms. Thus there are an even no. of vertices of odd degree

Eg - 

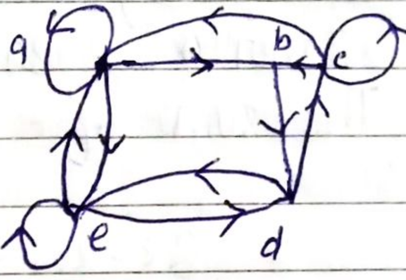

$deg(v_1) = 1$
$deg(v_2) = 2$

$deg(v_3) = 1$ | ~~Here total vertex = 3 (odd)~~

Here total vertex = 3 (odd)
Total degree = 4 (even)

Degree of vertex in directed graph:—

In-degree in graph G




$deg^-(a) = 3$
$deg^-(b) = 2$
$deg^-(c) = 2$
$deg^-(d) = 2$
$deg^-(e) = 3$

Out-degree in graph G

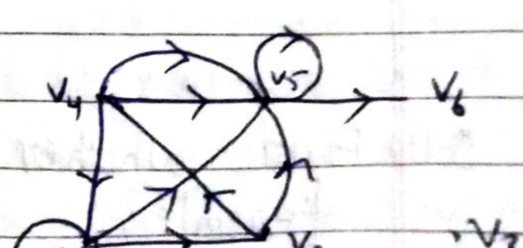
$deg^+(a) = 3$, $deg^+(b) = 1$, $deg^+(c) = 3$, $deg^+(d) = 2$
$deg^+(e) = 3$

<u>NOTE</u> :- Let G be a graph with directed edges
Then

$$\boxed{\sum_{v \in V} deg^-(v) = \sum_{v \in V} deg^+(v) = |E|}$$

Degree sequence of graph:— The degree of each vertex of graph write in an ascending order, the sequence so obtained is known as degree sequence.

Eg - $deg(v_1) = 5$, $deg(v_2) = 3$
$deg(v_3) = 0$ $deg(v_4) = 4$
$deg(v_5) = 7$ $deg(v_6) = 1$




$\therefore$ the degree sequence = $\langle 0, 1, 3, 4, 5, 7 \rangle$

Q. Prove the degree of a vertex in simple graph G with n vertices cann't be greater than (n-1)

Sol$^n$ Given that graph is simple so it has no loops & no $||^{rl}$ edges. Thus any vertex V can be adjacent to at most $(n-1)$ vertices.

Hence $0 \le deg(v) \le n-1$

Q. How many edges are there in graph with 7 vertex each of degree 4 ?

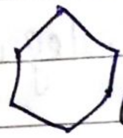
Ans- The sum of degree of vertices $= 4 \times 7 = 28$

$2e = 28 \Rightarrow e = 14$ so no. of edges $= 14$.

<u>Regular Graph :-</u> If every vertex of graph has same degree

Eg -

(i)

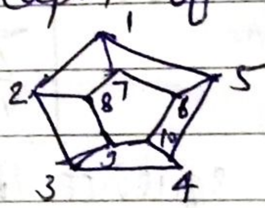


(ii)

(i) & (ii) degree of each vertex $= 2$

Q. Construct a 3-regular graph of 10 vertices

Ans Here $n = 3$




{ NOTE:- Size of n regular (s,t) graph $= \frac{ns}{2}$
where
$n =$ degree, $s =$ vertices, $t =$ edges

Q (i) Find whether 3 regular graph on 17 vertices in feasible?

(ii) Show that 3 regular graph on 14 vertices is feasible

(i) $S=17$, $n=3$

By size formula $t = \frac{ns}{2} = \frac{3\times17}{2} = 25.5$

The value of $t$ is not integer value so such a graph is not feasible.

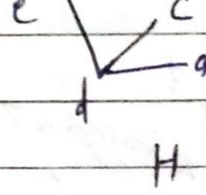
(ii) $S=14$ $n=3$

$t = \frac{14\times3}{2} = 21$

The value of $t$ is integer value, so such a graph is feasible.

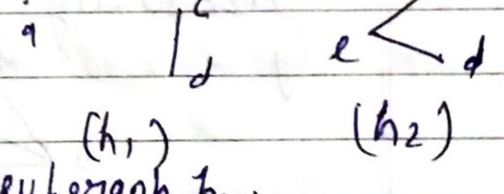
<u>Subgraphs:</u>— A graph h is said to be subgraph of graph H, if all vertices & all edges of h are in H & each edges h has same end vertices in h as in H.

eg-




H

subgraph:—

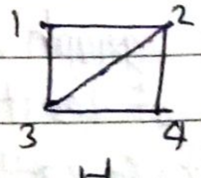



(h₁) (h₂)

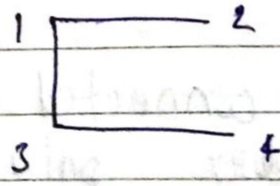
a & b are isolated in subgraph $h_1$.

<u>Spanning Subgraph:</u>— A subgraph h of graph H is called a spanning subgraph of graph H if h contains all vertices of H.

eg-




H




spanning subgraph.

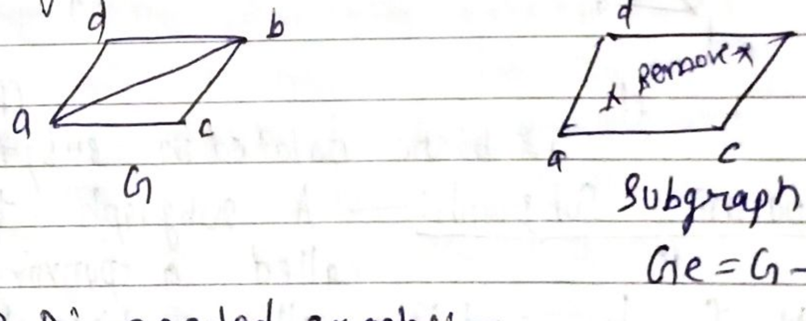
Disjoint Subgraph:— Let G be graph & $S_1(V_1, E_1)$, $S_2(V_2, E_2)$ be two subgraph of G. Then subgraphs are known as vertex disjoint if $V(S_1) \cap V(S_2) = \phi$ and are called edge disjoint if $E(S_1) \cap E(S_2) = \phi$

The vertex disjoint subgraph are always edge disjoint.

Induced Subgraph:— Suppose G be an undirected graph. let $v \in V$ the subgraph of G represented by $G-v$ has vertex set $V_1 = V - \{x\}$ and edge set $E_1 \subseteq E$ where E involves all edges in E except for those which are incident to vertex v. We conclude that the subgraph $G-v$ • induced by set $V_1$. $G-v$ cas also be written as $G_x$.

Subgraph Ge suppose that G be an undirected graph letting $e = (a, b) \in E$ we find subgraph denoted by Ge from G by deleting the edge e without removing the vertices.

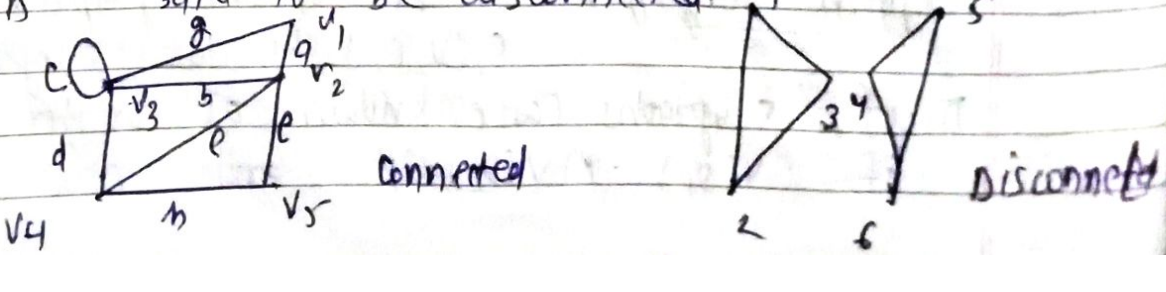
Eg —




Subgraph Ge

$Ge = G - e$

Connected & Disconnected graphs:— A graph G is said to be connected if there is at least one path b/w every pair of vertices in G otherwise it is said to be disconnected.



**Theorem:-** A simple graph (i.e. graph without $\parallel^{al}$ edges or self loops) with $n$ vertices & $k$ components can have at most $(n-k)\frac{(n-k+1)}{2}$ edges.

**Proof:-**

Component: A disconnected graph consists of 2 or more connected subgraphs. Each of these subgraph is called a component.

Suppose that the no. of vertices in each of the $k$ components of a graph $G$ be $n_1\ n_2 --- n_k$

Thus we have

$$n_1 + n_2 + --- + n_k = n$$

i.e. $$\sum_{i=1}^{k} n_i = n \quad \text{--- (1)}$$

where $n_i \geq 1$, $i = 1, 2 --- k$

consider

$$\sum_{i=1}^{k} (n_i - 1) = \sum_{i=1}^{k} n_i - \sum_{i=1}^{k} 1 = n-k$$

squaring both side we get

$$\left[\sum_{i=1}^{k} (n_i - 1)\right]^2 = (n-k)^2 = n^2 + k^2 - 2nk$$

or $$\sum_{i=1}^{k} (n_i^2 - 2n_i + 1) + \text{non-negative cross terms}$$

$$= n^2 + k^2 - 2nk \qquad [\because n_i - 1 \geq 0]$$

$$\Rightarrow \sum_{i=1}^{k} (n_i^2 - 2n_i + 1) \leq n^2 + k^2 - 2nk$$

$$\Rightarrow \sum_{i=1}^{k} (n_i^2) \leq n^2 + k^2 - 2nk + 2n - k = n^2 - (k-1)(2n-k)$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 \leq n^2 - (k-1)(2n-k) \quad \text{--- (2)}$$

Now the max. no. of edges in the $i^{th}$ component of $G$ (which is a simple connected graph) is

$\frac{1}{2} n_i (n_i - 1)$ . Thus the max. no. of edges in G is

$$\frac{1}{2} \sum_{i=1}^{k} (n_i - 1) n_i = \frac{1}{2} \left[ \sum_{i=1}^{k} n_i^2 - \sum_{i=1}^{k} n_i \right]$$

$$= \frac{1}{2} \sum_{i=1}^{k} n_i^2 - \frac{1}{2} n \quad (\text{using } 1)$$

$$= \frac{1}{2} [ n^2 - (k-1)(2n-k) ] - \frac{n}{2}$$

$$= \frac{1}{2} [ n^2 - 2nk + k^2 - 2n - k + n ]$$

$$= \frac{1}{2} [ (n-k)(n-k) + (n-k) ]$$

$$= \frac{1}{2} [ (n-k)(n-k+1) ]$$ Here the result.

→ Distance & Diameter in Graph

→ Complete Graph $k_n = {}^nC_2 = \frac{n(n-1)}{2}$

→ Cycles

→ Wheels

→ Bi partite Graphs

→ Complete Bipartite Graphs

→ Weighted graphs.

→ Shortest Path Problem ⇒

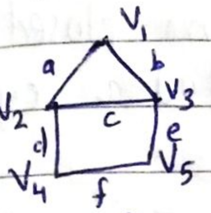
Dijkstra's Algorithm:—

→ Ring-sum of 2 Graph $(G_1 \oplus G_2)$

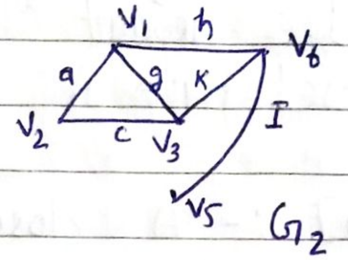
→ Complement Graph $(\bar{G})$ $\boxed{\frac{n(n-1)}{2} - e}$

→ Product of two Graph $(G_1 \times G_2)$

→ Difference of 2 Graph ($G_1 - G_2$) :– $G_1 - G_2$ is a graph having all those edges which are in $G_1$ but not in $G_2$.
$G_1 - G_2$ is also said to be complement of $G_2$ in $G_1$.




$G_1$



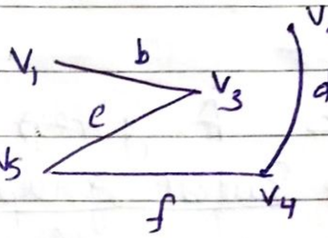

$G_2$

$\frac{n(n-1)}{2} - e$

$\rightarrow \frac{5 \times 4}{2} - 6$

$\Rightarrow$ 4 edge

$G_1 - G_2 \Rightarrow$




→ Decomposition of Graph $\Rightarrow$ $G_1 \cup G_2 = G$ & $G_1 \cap G = \text{null}$

→ Fusion of vertex :– It reduces no. of vertices by one but not alter the no. of edges.

→ Disjoint Graphs :– A graph having no vertex is common i.e. $V_1 \cap V_2 = \phi$

→ Isomorphic Graph :– If there exists a bijection mapping (one-one, onto)

(i) same no. of vertices.
(ii) same no. of edges.
(iii) Equal no. of vertices with given degree.
(iv) Adjacency relationship must be preserved.

→ Cutsets :– It is a set of edges such that removal of these edges produces a subgraph with more connected components.

→ Bridges → Cut vertex

PAGE NO. :

$N_0 \neq V_n$ $N_0 = V_n$

→ Walk → open walk → closed walk

→ Trail ⇒ no edge repeat (in open walk)

→ Path ⇒ no vertex repeat (in open walk)

→ length of path ⇒ no. of edges in a path

→ Circuit ⇒ No edge repeat (in closed walk)

→ Cycles ⇒ A closed path is called a cycle.

→ Euler Line [chain] :- A closed trail consisting of all edges of G.

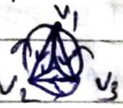
→ <u>Theorem</u> ⇒ A given connected graph G is an <u>Euler graph</u> if & only if all vertices of G are of even degree.

→ A connected multigraph has an Euler path but not an <u>Euler circuit</u> if & only if it has exactly two vertices of odd degree

→ Hamilton Path :- no vertex is repeated & no edge is repeated but it passes through all the vertices of G.

⇒ In complete graph with n-vertices there are $\left(\frac{n-1}{2}\right)$ edge disjoint Hamilton circuits, if n is an odd number ≥ 3

Complete Diagraph → complete symmetric diagraph





one edge directed from every vertex

→ complete Asymmetric diagraph

(one edge b/n every pair of vertices)

DATE : / /

PAGE NO. :

Euler's Formula :— $r$ (regions) $= e - v + 2$

Q. $v = 30$, $deg = 3$, $r = ?$
$n = 3$
$\Sigma deg(v) = 3 \times 30 = 90$
$90 = 2e \Rightarrow e = 45$
$r = 45 - 30 + 2 = 17$ regions.

$\Rightarrow$ $v$, $e (r, 2)$ then $3r \leq 2e$ & $e \leq 3v - 6$

→ Homeomorphic Graph 

→ Kuratowski's Two graphs ⇒ A complete & bipartite graph are non planar.

Kuratowski's Theorem :—
A graph is non planar if & only if it contains subgraph i.e. homeomorphic to either $K_5$ or $K_{3,3}$.

# Chapter - 5

## ~~Graph~~ GROUPS

→ Semigroups & monoids has useful applications in area of computer arithmetic, formal languages & study of FSM.
Group theory is used in coding theory.

$f :- A^n \longrightarrow A$ is called n-ary operation

$n=1 \Rightarrow f: A \rightarrow A$ (uniary)

$n=2 \Rightarrow f: A \times A \rightarrow A$ (binary)

Binary operation $\longrightarrow +, \cdot, o, *$

Properties of Binary operation :—

→ Commutative $a*b = b*a \quad \forall\ a, b \in A$

→ Associative $(a*b)*c = a*(b*c)$

- Semigroup :— $(S, *)$ is called a semigroup if the binary operation $*$ is associative in S.
- Monoid :— $(S, *)$ is said to be monoid if it satisfies closure, associativity & existence of identity element ($a*e = a = e*a$).

→ Closure property :— $x*y = (a,b)*(c,d) = (ac, bd)$

→ Associativity :— $x*(y*z) = (x*y)*z$

- Composition Table

- Addition modulo & Multiplication Modulo :—

$a +_m b = r \quad , \quad 0 \leq r \leq m$

where $r$ is least non negative remainder when $(a+b)$ is divided by $m$.

$a \times_p b = r \qquad 0 \leq r \leq p$

where $a \cdot b$ is divided by $p$.

eg- $15 +_5 7$ = remainder when $(15+7=22)$ is divided by 5
$= 2$ Ans

eg- $-23 +_3 3 = -23+3 = -20 = -21+1$ is divided by $3 = 1$

eg- $8 \times_5 3 = \frac{24}{5} = 4$ Ans

eg- $8 \times_5 (-3) = \frac{-24}{5} = \frac{-25+1}{5} = 1$ Ans

Groups :- closure, associativity, existence of identity & existence of inverse [$a*b=e=b*a$]

or

a monoid is said to be a group if each element of $S$ has its inverse in $S$.

Abelian or Commutative Group :— if $a*b = b*a$

Order of element :— $\boxed{a^n = e}$ $\quad o(a) = n$

Group Homomorphism :— Let $(G, *)$ & $(G', *')$ be 2 groups & let $f: G \rightarrow G'$ be a mapping, then $f$ is called a group homomorphism, if for all $a, b \in G$,

$f(a * b) = f(a) *' f(b)$

→ kernel of homomorphism

PAGE NO. :

→ Group Isomorphism :—

process to prove $G$ is isomorphic to $G'$

(i) Define $f: G \to G'$

(ii) Show $f$ is one-one $f(a) = f(b) \Rightarrow a^{-1} = b^{-1}$

(iii) Show $f$ is onto $f(b^{-1}) = (b^{-1})^{-1} = b$

(iv) Show $f$ is homomorphism $f(x * y) = f(x) * f(y)$

Permutation Group :—

If $S = \{1, 2, 3\}$ then

$P_3 = \{ P_1, P_2, P_3, P_4, P_5, P_6 \}$

Product of group :— $(G_1, *)$ $(G_2, *') \Rightarrow (G_1 \times G_2, *'')$

$(a_1, a_2) *'' (b_1, b_2) = (a_1 * b_1, a_2 *' b_2)$

identity element :—

$(a_1, a_2) *'' (e_1, e_2) = (a_1 * e_1, a_2 *' e_2) = (a_1, a_2)$

Existence of inverse :—

$(a_1, a_2) *'' (a_1^{-1}, a_2^{-1}) = (a_1 * a_1^{-1}, a_2 *' a_2^{-1}) = (e_1, e_2)$

CYCLIC GROUP ⇒ $g = a^n$

Q. $G = \{1, -1, i, -i\}$

$\Rightarrow \{i^4, i^2, i, i^3\}$ so $i$ is generator

Also,

$\{(-i)^4, (-i)^2, (-i)^3, (-i)^1\}$ so $-i$ is also generator

Coset :— $Ha = \{ha : h \in H\}$ is called right coset of $H$.
$aH = \{ah, h \in H\}$ is called left coset of $H$.

→ $H$ is subgroup of $G$ then $Hh = H = hH$

→ " (i) $Ha = Hb \Leftrightarrow ab^{-1} \in H$ (ii) $aH = bH \Leftrightarrow a^{-1}b \in H$

→ $Ha \cap Hb = \phi$ or $Ha = Hb$

& $aH \cap bH = \phi$ or $aH = bH$

Lagrange's Theorem :— The order of each subgroup of finite group is a divisor of order of group.

If $Ha_1, Ha_2 --- Ha_s$ are $s$ distinct right cosets of $H$ in $G$, then

$G = Ha_1 \cup Ha_2 \cup --- \cup Ha_s$

$\Rightarrow |G| = |Ha_1| + |Ha_2| + --- + |Ha_s|$

$O(G) = m + m + --- m$ (s time) $= ms$

$n = ms$

$s = \frac{n}{m} \Rightarrow m$ is divisor of $n$.

$O(H)$ is a divisor of $O(G)$

i.e. $= \frac{O(G)}{O(H)}$

Quotient Group :— (factor) $G$ be a group & $H$ be a normal subgroup of $G$ then $\frac{G}{H}$ of all right coset of $H$ in $G$ i.e. $\frac{G}{H} = \{Ha \mid a \in G\}$

$HaHb = Hab$

Rings :— A algebric structure $\langle R, +, \cdot \rangle$ is called a ring if

(i) $(R, +)$ is an abelian group
(ii) $(R, \cdot)$ is a semigroup
(iii) The operation '$\cdot$' is distributive are the operation '+'.

Ring with unity :— It has identity element w.r.t. multiplication composition

$$a \cdot e = a = e \cdot a$$

Commutative Ring :— $a \cdot b = b \cdot a$

→ zero divisor :— if $b \neq 0$, then $ab = 0$ or $ba = 0$

→ without zero :— If $ab = 0$ then $a = 0$ or $b = 0$

Cancellation Laws in a Ring :—

If $a \neq 0$, $ab = ac \Rightarrow b = c$
and $a \neq 0$ $ba = ca \Rightarrow b = c \quad \forall a, b, c \in R$

Integral Domain :— commutative ✓
has unit element ✓
without zero divisors ✓

→ Division Ring OR skew field :— unity ✓
each non zero element possesses multiplicative inverse. ✓

Isomorphism of Rings :—
$f(a+b) = f(a) + f(b)$
& $f(ab) = f(a) \cdot f(b)$
& f is onto