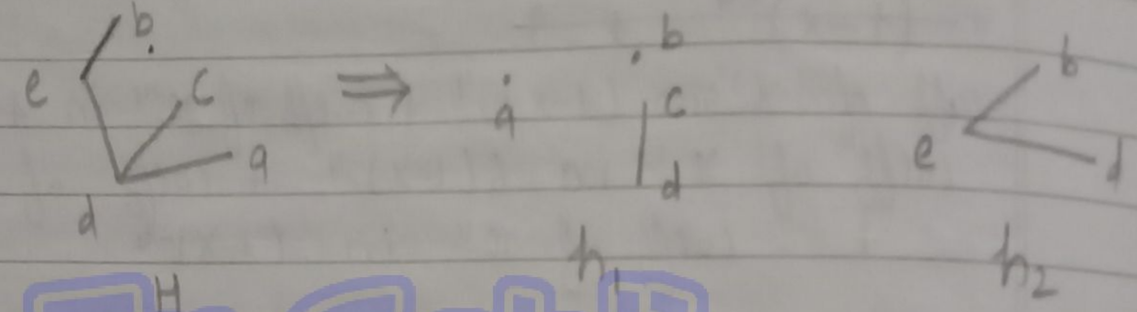
Q.1 Define the following terms by taking an Example

(i) Subgraph :— A graph h is said to be subgraph of graph H if all vertices & all edges of h are in H & each edges n has same end vertices in h as in H.

Eg —



(ii) Spanning Subgraph :— A subgraph h of graph H is called a spanning subgraph of graph H if h contains all vertices of H.

Eg -

(iii) Complete graph :— The complete graph on n vertices denoted by k_n is the simple graph that contains exactly one edge b/w each pair of distinct vertices

$$\boxed{\text{The size of } k_n = {}^{n}C_2 = \frac{n(n-1)}{2}}$$

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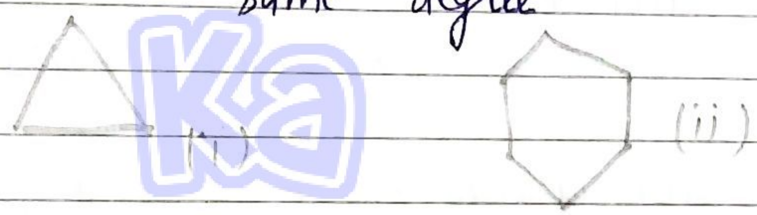
(iv) **Isomorphic graph :—** Two graph $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to be isomorphic to each other if there exists a bijection mapping f from V_1 to V_2 i.e. $f : V_1 \to V_2$ such that for each of the vertices The function f is called an isomorphic from G_1 to G_2 or in simple language,

(i) same no. of vertices (ii) same no. of edges
(iii) An equal no. of vertices with given degree
(iv) Adjacency relationship must be preserved.

(v) **Regular Graph :—** If every vertex of graph has same degree

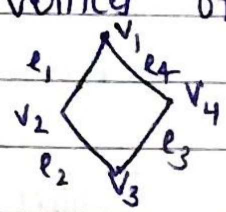
Eg —



(vi) **Eulerian chains :—** In a graph G, a closed trail consisting of all edges of G is called "Euler line". An Euler line passes through all edge of G but not through all the vertices of G in case some of vertices of G be isolated.

(vii) **Hamiltonion cycle :—** In a graph G, a path is said to be Hamilton path if it passes through all the vertices of G.

Eg — $W = V_1 e_1 V_2 e_2 V_3 e_3 V_4 e_4 V_1$



(viii) **Diagraph :—** A diagraph is short for directed graph and it is a diagram composed of points called vertices

(ix) Walk, path & Circuit ⇒

Path :- An open walk in which no vertex appears more than once is called path.

Walk :- A finite alternating sequence of vertices and edges denoted by W begining & ending with vertices.

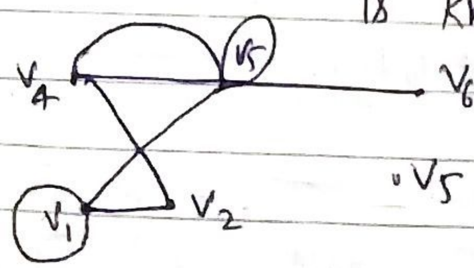
Circuit :- A closed walk in which no edge appears more than once is called a circuit.

(x) Euler path :- A closed trail consisting of all edges of G.

(xi) Euler circuit ⇒ A closed path is possible only when each vertex is of even degree.

(xii) Pendant vertex :- A vertex of degree one is known as pendant vertex

Eg -



$deg(V_6) = 1$
V_6 is pendent vertex.

Q(2) Define following Operation on graph

(i) Union of sum Graph :- Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$

then

composed of points called vertices

(ix) Walk, path & Circuit ⇒

Path :- An open walk in which no vertex appears more than once is called path.

Walk :- A finite alternating sequence of vertices and edges denoted by w begining & ending with vertices.

Circuit :- A closed walk in which no edge appears more than once is called a circuit.

(x) Euler path :- A closed trail consisting of all edges of G.

(xi) Euler circuit ⇒ A closed path is possible only when each vertex is of even degree.

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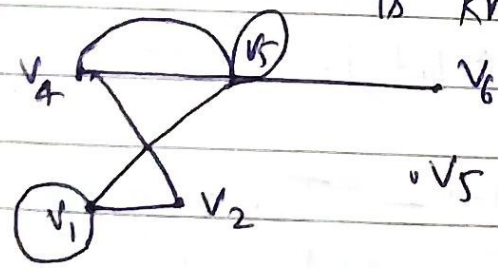
Eg -



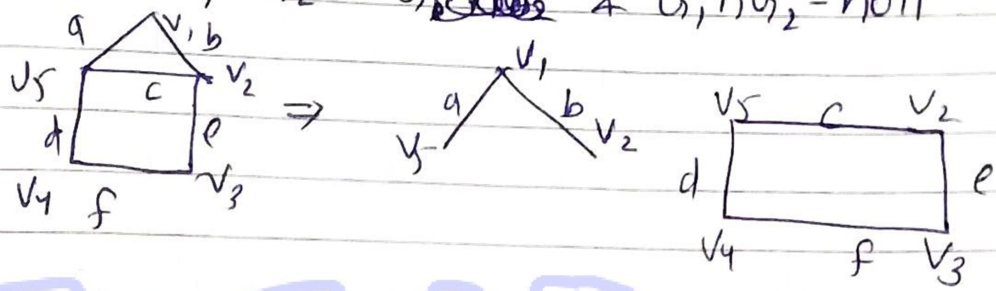
$deg(v_6) = 1$
v_6 is pendent vertex.

Q(2) Define following Operation on graph

(i) Union of sum Graph :- Let $G_1 = (V, E_1)$ and $G_2 = (V_2, E_2)$ then if $V = V_1 \cup V_2$ & $E = E_1 \cup E_2$ & is denoted by $G_1 \cup G_2$

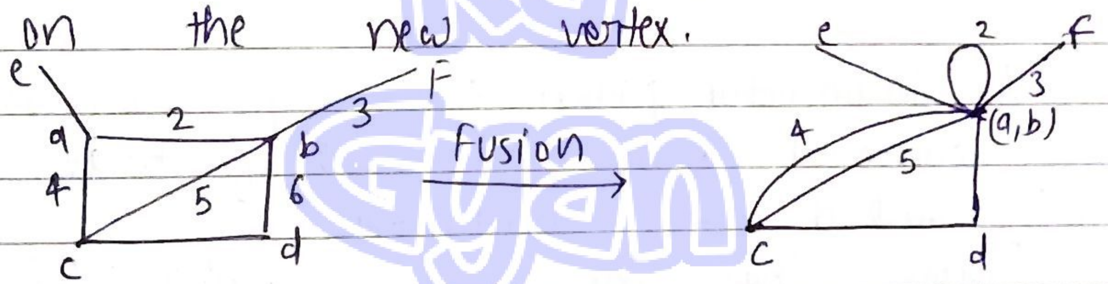
(ii) Decomposition of Graph :— A graph is said to be decomposed into 2 subgraph G_1 & G_2 if $G_1 \cup G_2 = G$ & $G_1 \cap G_2 =$ null

Eg -



(iii) fusion of vertices :— A pair of vertices V_1 in a graph are said to be fused if the two vertices are replaced by a single New vertices such that every edge that was incident on either V or V on both is incident on the new vertex.

Eg -



Q.3 Explain Handshaking theorem.

Ans - The sum of degree of all vertices in graph is equal to twice the no. of edges in the graph.

Proof :- Let $G = (V,E)$ be graph with n vertices & no. of edges are $|E| = e$. Since each edge is incident with 2 vertices so each edge is contributed a count of 1 to each deg(v) & deg(u). Edges will contribute 2e degree for all vertices.

Thus $\boxed{\sum_{i=1}^{n} Deg(v_i) = 2e = 2|E|}$

Q.4 Show that a 3-regular graph on 14 vertices is feasible.

Ans - $S = 14$, $n = 3$

Now, by using size formula $t = \frac{ns}{2}$

$$t = \frac{14 \times 3}{2} = 21$$

Value of t is an integral value so such graph is feasible.

Q.5 Prove that a simple graph i.e a graph without parallel edges or self loop with n vertices & k components can have at $\frac{(n-k)(n-k+1)}{2}$ edges

Ans - A disconnected graph consists of 2 or more connected subgraphs. Each of these subgraph is called a component.

Suppose that the no. of vertices in each of the k components of graph G be $n_1, n_2 \ldots n_k$

Thus we have

$$n_1 + n_2 + n_3 \ldots + n_k = n$$

i.e. $\sum_{i=1}^{k} n_i = n$ —— (1)

where $n_i \geq 1$, $i = 1, 2 \ldots k$

consider

$$\sum_{i=1}^{k} (n_i - 1) = \sum_{i=1}^{k} n_i - \sum_{i=1}^{k} 1 = n - k$$

Squaring both side

$$\left[\sum_{i=1}^{k} (n_i - 1)\right]^2 = (n-k)^2 = n^2 + k^2 - 2nk$$

or $\sum_{i=1}^{k} (n_i^2 - 2n_i + 1)$ + non negative cross terms

$= n^2 + k^2 - 2nk \quad (\because n_i - 1 \geq 0)$

$\Rightarrow \sum_{i=1}^{k} n_i^2 \leq n^2 + k^2 - 2nk + 2n - k = n^2 - (k-1)(2n-k)$

$\Rightarrow \sum_{i=1}^{k} n_i^2 \leq n^2 - (k-1)(2n-k)$ —— (2)

Now max. no of edge in the ith component of G is $\frac{1}{2} n_i (n_i - 1)$.

Thus max. no. of edges in G is

$\frac{1}{2} \sum_{i=1}^{k} (n_i - 1) n_i = \frac{1}{2} \left[\sum_{i=1}^{k} n_i^2 - \sum_{i=1}^{k} n_i \right]$

Using (1) $\frac{1}{2} \sum_{i=1}^{k} n_i^2 - \frac{1}{2} n$

$= \frac{1}{2} (n^2 - (k-1)(2n-k)) - \frac{1}{2} n$

$= \frac{1}{2} [n^2 - 2nk + k^2 + 2n - k - n]$

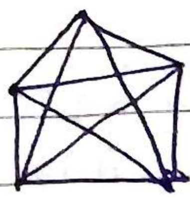
$= \frac{1}{2} [(n-k)(n-k) + (n-k)]$

$= \frac{1}{2} (n-k)(n-k+1)$ H.P.

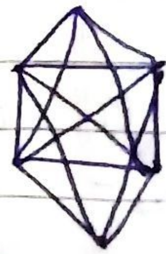
Q.(6) Sketch the graph k_n w_n & c_n when $4 \leq n \leq 8$

Ans - (i) k_4

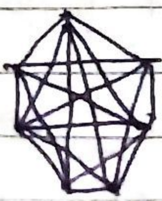
(ii) k_5



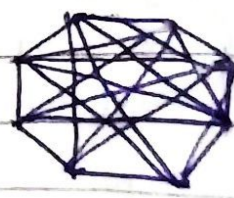
(iii) k_6



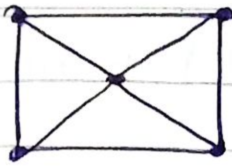
(iv) k_7



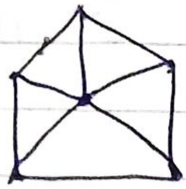
(v) k_8



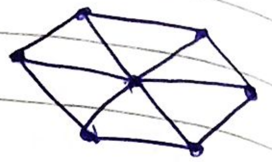
For W_4 :-



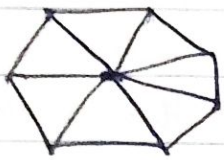
W_5 :-



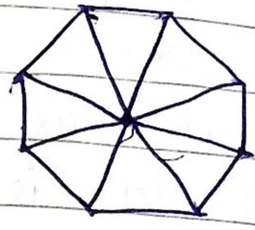
W_6 :-



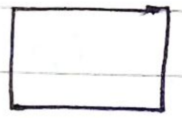
W_7 :-



W_8 :-

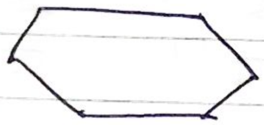


For C_4 :-

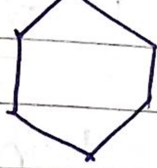


C_5 :-

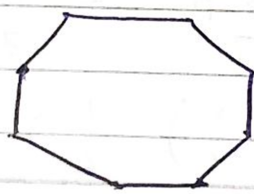
C_6 :-



C_7 :-



C_8 :-



Q. 7 Display the complete bipartite graph K_{23}, K_{33} K_{35} & K_{26}

Ans -

K_{23}

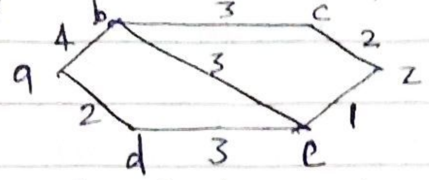
$K_{3,3}$

$K_{3,5}$

$K_{2,6}$

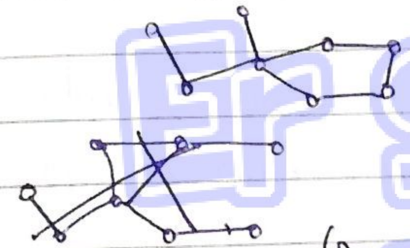
Q. 8 Explain Dijkrstra's Algorithm with example.

Ans - Dijkestra's Algo proceeds by finding the length of the shortest path from a to the second vertex and so on. Until the length of the shortest path from a to z is found for - what is the length of shortest path b/w a to z.



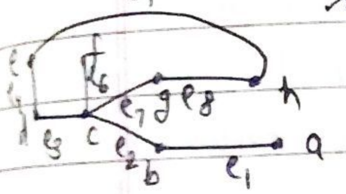
Sol^n the shortest path can early found by injection. we come across certain idea useful is understanding Dijkastra's algorithm.

Q 9 Which one is isomorphic



Ay - (b)

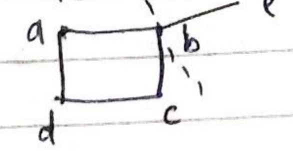
Because $V(G_1) = V(G_2)$
$E(G_1) = E(G_2)$
$Deg(G_1) = Deg(G_2)$



So (b) is isomorphic

Q.10 Define with example

(i) Cutsets vertex :— In separable graph a vertex whose removal disconnect; the graph is called a cut-vertex



(ii) Bridges :- If a cutset for a connected graph consists of exactly one edge then the edge is called a bridges for graph.

(iii) Chromatic Number :— Pointing all vertices of graph with edges in such a way the two adjacent vertices an coloured with different colors is called proper coloring a vertex X coloring of a graph.

(iv) Euler's formula:- Let G be a connected planes simple graph with e edges & v vertices is r be the no. of origins in planar graph representation of G, then

$$\boxed{r = e - v + 2}$$

(v) Homographic graph:- Two graph are will to be homographic graph can be obtained from the other by Creater of edges is satisfied in all graph with merge of edge Euler.

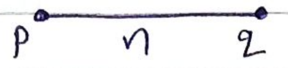
(vi) kurtowski's ⟹ A graph is non planar if & only if it contains subgraph that is homorgraphic to either K_5 or K_{33}.

(vii) Cycles:- A closed path is called cycles.

(viii) Planar graph:- A graph is called planar if it can be trail in the plane such that no two of edges intersects each other except of vertex.

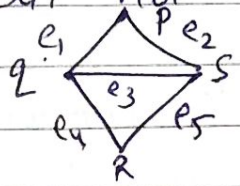
Q(11) (a) Draw following graph
Neither Eulerian or Hermiltonian

Ans –

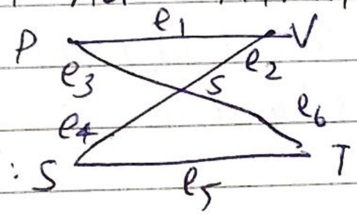


The graph G cann't have any closed walk without repeating the vertices & edges.

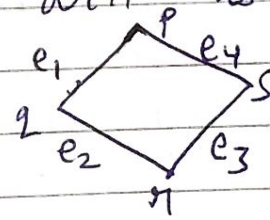
(b) Hamilton but not Euler



(c) Euler but not Hamilton



(d) Eulerion as well as Hamiltonian



$w = pe_1qe_2re_3se_4p$

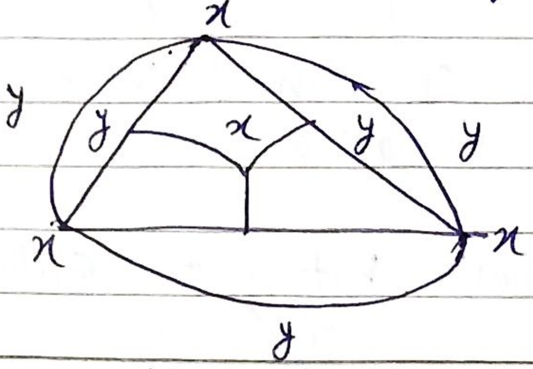
Q.12 Show that isomorphism of simple graph is an equivalence relation.

Ans – Reflexive:- Let G be any simple graph than G is isomorphic to itself by the identity functions

Symmetric:- Let G is isomorphic to another simple graph is then there exist a one to one correspondance F from G to H.

Transitive:- There exist bijection g from G to H & from H to K that preserve adjacency & non adjacency.

Thus, isomorphism is an equivalence.



Q. 13 Prove that simple connected (p, q) graph G is Hamiltonian if $q \geq \frac{p^2 - 3p + 6}{2}$

Ans — Let U & V be two adjacent vertices of G. Let H be the subgraph of G, obtained key removing U & V, and all edges incident on V clearly then

$$B(H) = q - Deg(U) - Deg(V)$$

The max edges in H can be ${}^{p-2}C_2$

$$= \frac{p^2 - 5p + 6}{2}$$

$$= \quad Deg(V) + Deg(U) \geq q - \frac{p^2 - 5p + 6}{2}$$

$$\geq \frac{p^2 - 3p + 6}{2} - \frac{p^2 - 5p + 6}{2} - p$$

G is Hamiltonian

Q. 14 Let a connected planar graph has 20 vertices each of degree 3 into how many regions does on representation of this planar graph split the plane.

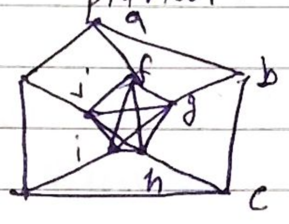
Ans- $v = 20$ and $\deg = 3$
sum of edges of vertice $3v = 3 \times 20 = 60$
is equal to twice no. of edge $2e$
we have $2e = 60$

$e = 30$

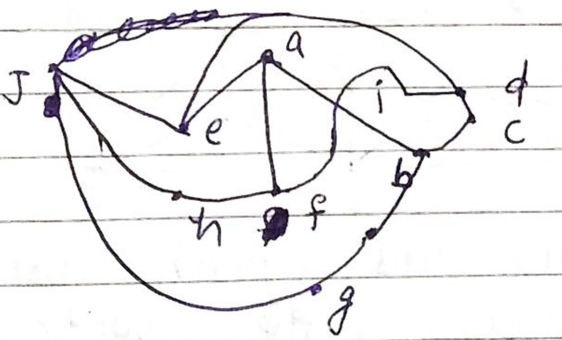
so $r = e - v + 2$

$r = 30 - 20 + 2 = 12$ Ans

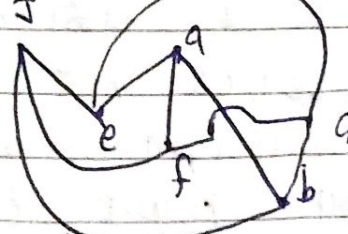
Q. (15) show that petarson graph given in following fig. is non planar.



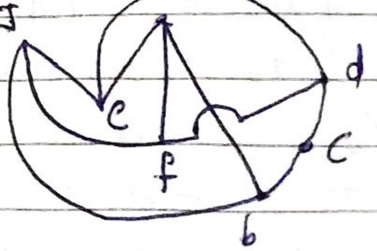
Ans- The given graph is a subgraph of petrson graph since it involves all vertices but edges are less than petrson graph



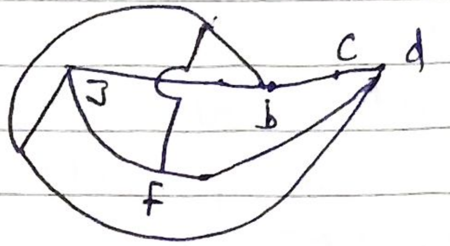
This graph is obtained from K_{33} by 4 sub division a displayed in following figure.



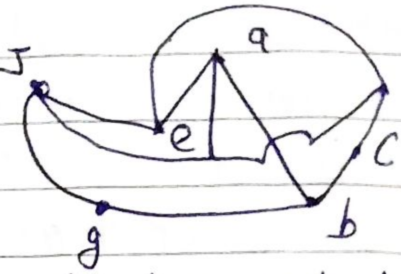
(i) K_{33}



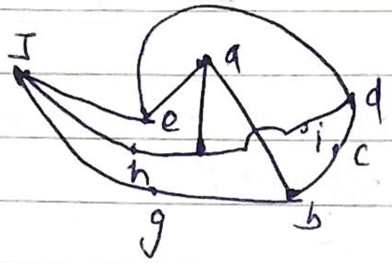
(ii) One sub division



(iii) Two sub division



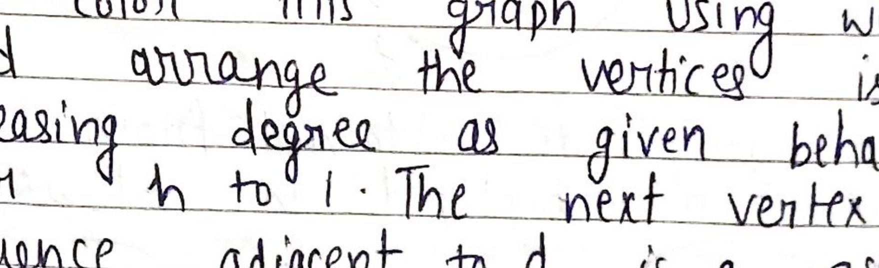
(iv) 3 sub division



(V) 4 sub division

graph (1) & (5) is subgraph of petarson
Hence theorem of kytowski is petarson
graph is non-planar.

Q.16 Point the graph with proper colors & find the chromatic no. in following graph.

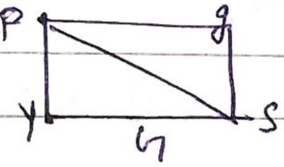


Sol^n – We color this graph using warn power method arrange the vertices is order to decreasing degree as given behave assign color h to 1. The next vertex in the sequence adjacent to d is a assign color C_1 to a then next vertex not adjacent to i and a is d, assign color C_1 to d – next take second color C_2, assign C_2 to b and so on.

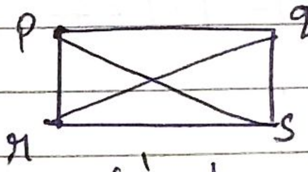
vertex	i	b	c	h	i	e	j	g	a	d
Degree	7	6	6	4	4	3	3	3	2	2
color	C_1	C_2	C_3	C_3	C_2	C_4	C_4	C_4	C_1	C_1

So minimum four color $C_1, C_2\ C_3\ C_4$ are needed to point the graph properly therefore the chromatic number $\tau(U) = 4$.

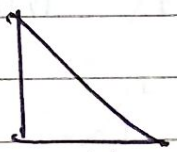
Q.(17) Obtain chromatic polynomial chromatic no. & no. of ways of proper coloring with max. color for the following graph $G(V,E)$.



Sol^n- The given graph G is subgraph of K_4 Add an edge (r,q) to G to make it K_4. Now we find graph G_e' & G_e'' follow



$G_1' = K_4$



$G_1'' = K_3$

Therefore, the given graph is decomposed into K_4 & K_3, the chromatic polynomial of which the known.

$$P(G_1', h) = h(h-1)(h-2)(h-3)$$

& $$P(G_1'', h) = h(h-1)(h-2)$$

Since, we have

$$P(G,h) = P(G_1',h) + P(G_1'',h)$$
$$= h(h-1)(h-2)(h-3) + h(h-1)(h-2)$$
$$= h(h-1)(h-2)(h-3+1)$$
$$= h(h-1)(h-2)^2$$

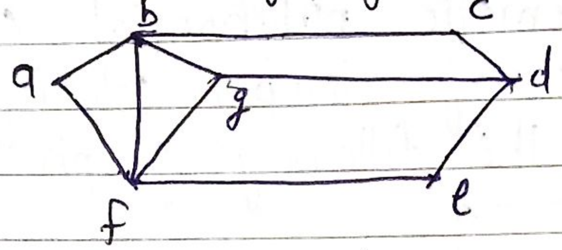
which required chromatic polynomial the least value of h for which $p(G,h) > 0$ is 3

$S(G) = 3$

and the no. of ways of proper coloring with 3 edges in $\lfloor 3 = 6$ Ans

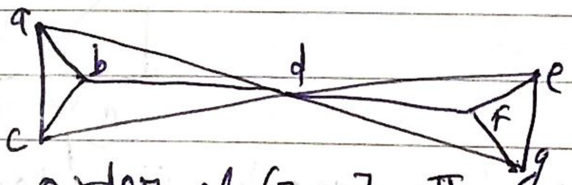
Q. (18) Determines the values of $\alpha(G)$, $\beta(G)$, $d(G)$ & $p(h)$ for following graph

(i)



Ans - The min. vertex can for G is the set
$k = \{b, d, f\}$
Hence, the vertex $\beta(G) = |k| = 3$
Since, the max. vertex independent set per G is set $I = \{a, c, e, g\}$ Thus vertex independence no. $\alpha(G) = |I| = 4$

(ii)



Here order of $G = 7$ Thus $p(G) = [7/2] = 3$
Since $s(a,c), (b,d), (e,g)$ 3 is edge independent so $B(G) = 3$
we know that $\alpha(G) + B(G) = n$
$\Rightarrow \alpha(G) = n - B(G) = 7 - 3 = 4$
$\alpha(G) = 4$
Further one of maximum independent sets of G is $I = \{e, g\}$
So $\alpha(G) = |I| = 2$
$\alpha(G) + B(G) = n$

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$$B(G) = n - \alpha(G)$$

$$B(G) = 7 - 2 = 5$$

$$B(G) = 5 \text{ Ans}$$

—— x ——

GROUPS

Q.1 Find the difference b/w the semigroup, monoid and abelian group with suitable example.

Ans- **Semigroup:-** An algebra structure (S, U) is called a semigroup if the binary operation is associative in e. and closure.

Monoid:- A algebric structure is said to be monoid if it is

(i) closure
(ii) Associative
(iii) Existence of identify element

Abelion Group:- A algebric structure is said to be abelian group if it is

(i) Closure
(ii) Associative
(iii) Existence of identify element
(iv) existence of inverse.

Eg -

$$a * b = \frac{ab}{2}$$

Closure:- let $a, b \in G$

$a * b$ are non zero real no.
ab are non zero real no.
$ab/2$ is also " " "

$\frac{ab}{2} \in G = (a * b) \in G$

G is closure.

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Associativity:—

$a*(b*c) = a*\frac{bc}{2} = \frac{a}{2}\left(\frac{bc}{2}\right) = \frac{abc}{4}$

$= \left(\frac{ab}{2}\right)\left(\frac{c}{2}\right) = (a*b)\frac{c}{2} = (a*b)*c$

$\Rightarrow \quad a*(b*c) = (a*b)*c \quad \forall\ a,b,c \in G$

so, associativity holds in G.

Existence of identity:— Let e be the identity element in G then

$a*e = a \quad , \quad \forall\ a \in G$

$= \frac{ae}{2} = a$

$e = 2 \in G$

2 is the identity element in G.

Since a is a arbitrary element so each element of G has its inverse in G.

Q.2 Let S be an non empty set with operation $a*b = a$. Show that S is semigroup but not monoid

Ans— Given $a*b = a$ —— (1)

Closure:— Let $a, b \in S$ then $a*b = a \in S$

So S is closure

Associativity:—

$a*(b*c) = a*b$

using (1)

$= a$ —— (2)

$(a*b)*c = a*c = a$ —— (3)

from eqn — (2) & (3) $a*(b*c) = (a*b)*c$

So S is semigroup.

Further suppose 'e' be the identity element

$a * e = a = e * a \quad , \forall \; a \in S$

Since $a * e = a$ but $e * a = e + a \quad \forall \; a \in S$

e exist in the set S where satisfies the property of identity element.

Hence S is not monoid.

Q.③ Let $\{x, y\}$ be semigroup where $x \cdot x = y$ then show (i) $x \cdot y = y \cdot x$ (ii) $y \cdot y = y$

Ans– (i) Given :– $x \cdot x = y$ — ①

$= x \cdot y = x \cdot (x x)$

$= (x \cdot x) x$

$= y \cdot x$

(ii) To show $y \cdot y = y$

since set $\{x, y\}$ is closed under the operation so we have 2 option

$x \cdot y = x$ or $x \cdot y = y$

If $x \cdot y = x$ then

LHS $y \cdot y = y \cdot (x x)$

$= (y \cdot x) x = (x \cdot y) \cdot x = x \cdot x = y = RHS$

If $x \cdot y = y$ then

$= y \cdot y$

$= (x \cdot x) \cdot y$

$x \cdot (x \cdot y)$ [By associativity]

$= x \cdot y$

$= y = RHS$

Q.④ Show $a * b = \frac{ab}{2}$ is an abelian group where G is non zero real no.

Ans – Given $a * b = \frac{ab}{2} \quad \forall \; a, b \in G$

a & b are non zero real no.
ab is also " "
$ab/2$ is " " ".

$$\frac{ab}{2} \in G = a * b \in G$$

G is closed.

Associativity:—

$$a*(b*c) = \frac{abc}{4} \quad (1)$$

$$(a*b)*c = \frac{abc}{4} \quad (2)$$

So $a*(b*c) = (a*b)*c$

Existence of identity :— let e is a identity element in

$$a*e = a$$

$$= \frac{ae}{2} = a \Rightarrow e = 2 \in G$$

Existence of inverse:—
Let b be the inverse of $a \in G$ then $a*b = 2$

$$\frac{ab}{2} = 2 \Rightarrow b = \frac{4}{a} \in G \quad (a \neq 0)$$

Since a is a arbitary element so each element of G has inverse is G.

Abelian Group

let $a, b \in G$ then

$$a*b = \frac{ab}{2} = \frac{ba}{2} = b*a$$

$$= a*b = b*a \quad \forall a,b \in G$$

there $(G,*)$ is a abelian Group.

Q ⑤ Show that the Group $K_4 = \{e, a, b, c\}$ is an abelian group the operation defined by the following table

$\cdot$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Ans - Given $K_4 = \{e, a, b, c\}$

Closure :- Since all element in table are element of K_1 to K_4 is closed.

Associativity :- It can be easily verified that operation is associative.

Identity Element :- first row coincides with top row

Existence Inverse :- $e \cdot e = e \Rightarrow e^{-1} = e$

$a \cdot a = e \Rightarrow a^{-1} = a$

$b \cdot b = e \Rightarrow b^{-1} = b$

$c \cdot c = e = c^{-1} = c$

So the K_4 is abelian.

Q.⑥ Find order of each element is graph $(G, *)$ where $G = \{1, -1, i, -i\}$

Ans - Here identity element $e = 1$

Now $(1)^1 = 1 \Rightarrow O(1) = 1$

$(-1)^2 = 1 \Rightarrow O(-1) = 2$

$(i)^4 = 1 \Rightarrow O(i) = 4$

$(-i)^4 = 1 \Rightarrow O(-i) = 4$

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Q.7 Show that group $G = \{Z, +)$ is isomorphic to the group $G' = (nz, +)$ where n is a given integer.

Ans - Let us define a mapping

$f: G \to G$ such that $f(a) = na \quad \forall\, a \in Z$

one-one :— $f(a) = f(b)$

$na = nb \Rightarrow a = b$

$\therefore$ f is one-one

onto :— let $b' \in G'$ then there exist $n \in Z$ such that $b = nm$

$f(z) = nm = b, \; n \in Z$

each element of G' has its primary in G.

$\therefore$ f is onto.

Homomorphism :— Let $a, b \in G$ then

$f(\alpha + \beta) = n(a+b) = na + nb$

$= f(a) + f(b)$

f is homomorphism

Here f is a isomorphic.

Q.8 Define Cyclic Group & show that $G = (\{0, 1, 2, 3, 4, 5\}, +_6)$ is cyclic. Find the generators.

Ans - A group G is said to be cyclic if there exists atleast one element $a \in G$, such that every element of G can be written as an integral power of a i.e. for each $g \in G \; \exists \; n \in Z$ such that $g = a^n$

The element a is called the generator of cyclic group G and then G can be written as

$G = [a]$

$G = (\{0, 1, 2, 3, 4, 5\}, +_6) \Rightarrow O(G) = 6$. If there exists are element $a \in G$. such that $o(a) = O(G) = 6$ then G must be cyclic & a is the generator of G. Also identity element of G is 0.

Now $O(0) = 1$

$1 +_6 1 +_6 1 +_6 1 +_6 1 +_6 1 = 0 \Rightarrow O(1) = 6$

$2 +_6 2 +_6 2 = \quad O(2) = 3$

$3 +_6 3 = \quad O(3) = 2$

$4 +_6 4 +_6 4 = \quad O(4) = 3$

$5 +_6 5 +_6 5 +_6 5 +_6 5 +_6 5 = O(5) = 6$

So, there are 2 elements 1 & 5 in G such that

$$O(G) = O(1) = O(5) = 6$$

Hence G is cyclic and 1, 5 are generators of G.

Q.9 Define following terms—

Sol^n–

(i) Coset :– let G be group & H be any subgroup of G. Also suppose a be any element of G. Then the set

$$Ha = \{ha : h \in H\}$$ is called right coset of H in G generated by a.

(ii) Normal Subgroup :– It is a subgroup that is invariant under conjunction by members of the group of which it is a part.

(iii) Quotient Group :– It is a group for the operation multiplication of cosets defined by $HaHb = Hab$. This group G/H is called quotient group of G by H.

(iv) Zero divisors :– A non zero element 'a' of a ring R is called a zero divisor if there exists an element $b \neq 0 \in R$ such that either $ab = 0$ or $ba = 0$

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(v) Rings:— An algebric structure $<R, +, \cdot>$ is called a ring if
(a) $(R, +)$ is abelian group
(b) $(R, \cdot)$ is semigroup
(c) '$\cdot$' is distributive over '+'.

(vi) Integral Domain:— A ring is called an integral domain if it (i) is commutative
(ii) has unit element
(iii) is without zero divisors

(vii) Field:— A ring R with atleast two elements is called a field if it,
(i) is commutative
(ii) has unity
(iii) is such that each non-zero element possesses multiplicative inverse.

Q.10 Let R be a congerence relation on group $(G, *)$ then the quotient set G with operation defined by $[a] \circledast [b] = [a*b]$ is a group.

Solⁿ— Here $\frac{G}{R} = \{[a] : a \in G\}$

closure:— let $[a], [b] \in \frac{G}{R}$ then

$$[a] \circledast [b] = [a*b] \in \frac{G}{R}$$

$\therefore$ G/R is closed

Associativity:— Let $[a], [b], [c] \in \frac{G}{R}$ then

$([a] \circledast [b]) \circledast [c] = [a * b] \circledast [c]$

$= [(a * b) * c]$

$= [a * (b * c)]$

$= [a] \circledast [b * c]$

$= [a] \circledast ([b] \circledast [c])$

$\therefore$ Associativity holds in $\frac{G}{R}$

Identity element :- $e \in G \Rightarrow [e] \in \frac{G}{R}$

Let $[a] \in \frac{G}{R}$ then

$[a] \circledast [e] = [a * e] = [a]$

and $[e] \circledast [a] = [e * a] = [a]$

$\because [a] \circledast [e] = [e] \circledast [a]$

$\Rightarrow [e]$ is the identity in $\frac{G}{R}$

Existence of inverse:- Let $a \in G \Rightarrow a^{-1} \in G$

$\Rightarrow [a^{-1}] \in \frac{G}{R}$

Let $[a] \in \frac{G}{R}$ then

$[a] \circledast [a^{-1}] = [a * a^{-1}] = [e]$

and $[a^{-1}] \circledast [a] = [a^{-1} * a] = [e]$

$\Rightarrow [a] \circledast [a^{-1}] = [e] = [a^{-1}] \circledast [a]$

$\Rightarrow [a^{-1}]$ is the inverse of $[a]$

Hence, each element of $\frac{G}{R}$ has its inverse in $\frac{G}{R}$

Therefore, $\frac{G}{R}$ is a group for operation $\circledast$

Q11 The set $R = \{0, 1, 2, 3, 4, 5\}$ is a commutative ring w.r.t. '$+_6$' & '$\times_6$' as two Ring compositions.

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Sol^n We form a composition table for 'R' for the compositions '$+_6$' & '$\times_6$' as:

$+_6$	0	1	2	3	4	5
$\boxed{0}$	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

$\times_6$	0	1	2	3	4	5
0	0	0	0	0	0	0
$\boxed{1}$	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

From the composition tables it is clear that

(i) $(R, +)$ is an abelian group with identity 0.
Also $0^{-1} = 0$, $1^{-1} = 5$, $2^{-1} = 4$, $3^{-1} = 3$, $4^{-1} = 2$, $5^{-1} = 1$

(ii) R is closed w.r.t the composition $\times_6$.

(iii) Also we know that '$\times_6$' is an associative composition in R ie.
$a \times_6 (b \times_6 c) = (a \times_6 b) \times_6 c \ \forall \ a, b, c \in R$

(iv) Let $a, b, c \in R$
$a \times_6 (b +_6 c) = a \times_6 (b + c)$
= least non-negative remainder when $a \cdot (b+c)$ is divided by 6.
= least non-negative remainder when $ab + ac$ is divided by 6.
$= (ab) +_6 (ac)$
$= (a \times_6 b) +_6 ac$
$= (a \times_6 b) +_6 (a \times_6 c)$

Similarly, we can prove that

$(b +_6 c) \times_6 a = (b \times_6 a) +_6 (c \times_6 a)$

Hence multiplication distributes over addition.

(V) Also from the table it is clear that

$(a \times_6 b) = (b \times_6 a)$

$\therefore \langle R, +_6, \times_6 \rangle$ is a commutative ring.

(vi) Also $1 \in R$ is identity element for composition $\times_6$.

$\therefore$ R is a commutative ring with unity.

Q (12) Let $S = \{1, 2\}$ & $R = P(S)$ where $P(S)$ is power set of S i.e. $R = \{\phi, \{1\}, \{2\}, \{1,2\}\}$. Define the operation '+' & '·' on R by

$A + B = A \nabla B = \{n \mid n \in A \text{ or } n \in B \text{ but not both}\}$

$A \cdot B = A \cap B$ = intersection of A & B

Prove that $(R, +, \cdot)$ is a ring without zero divisors.

Sol^n – The composition tables can be constructed as follows: —

+	ϕ	$\{1\}$	$\{2\}$	$\{1,2\}$
ϕ	ϕ	$\{1\}$	$\{2\}$	$\{1,2\}$
$\{1\}$	$\{1\}$	ϕ	$\{1,2\}$	$\{2\}$
$\{2\}$	$\{2\}$	$\{1,2\}$	ϕ	$\{1\}$
$\{1,2\}$	$\{1,2\}$	$\{2\}$	$\{1\}$	ϕ

·	ϕ	$\{1\}$	$\{2\}$	$\{1,2\}$
ϕ	ϕ	ϕ	ϕ	ϕ
$\{1\}$	ϕ	$\{1\}$	ϕ	$\{1\}$
$\{2\}$	ϕ	ϕ	$\{2\}$	$\{2\}$
$\{1,2\}$	ϕ	$\{1\}$	$\{2\}$	$\{1,2\}$

From table (1) it is obvious that $(R, +)$ is abelian group.

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From table 2 it is clear that '.' is binary & associative in R. Also table is symmetric about the leading diagonal so the '.' is commutative further $\{1,2\}$ is a unit element for '.'.

Hence R is commutative ring with unity.

Again from table (2),

$$\{1\} \cdot \{2\} = \phi$$

where $\{1\} \neq \phi$, $\{2\} \neq \phi$

Thus,

$(R, +, \cdot)$ is a ring without zero divisors.