4E1213

4E1213

B.Tech. (Sem.IV) (Main) Examination, May-2019
Computer Science and Engineering
4CS2-01 Discrete Mathematics Structure

Time : 3 Hours

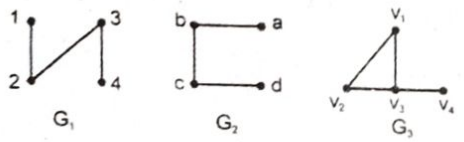
Maximum Marks : 120

Instructions to Candidates :
Attempt all ten questions from Part A, five questions out of seven questions from Part B and four questions out of five from Part C. Schematic diagrams must be shown wherever necessary. Any data you feel missing may suitably be assumed and stated clearly. Units of quantities used/calculated must be stated clearly.

PART - A

(Answer should be given up to 25 words only). All questions are compulsory. **(10 × 2 = 20)**

1. Prove that for any two sets A and B : $A - (A \cap B) = A - B$.
2. Give an example of a partially ordered set which is not a lattice.
3. Show that the multiplicative group $G = \{1, -1, i, -i\}$ is cyclic. Find its generators.
4. Define finite state Machines.
5. Find the minimum number of students in a school to be sure that 5 of them are born in the same month.
6. Prove that α^2 is an even integer, then α is an even integer.
7. Find the generating function for the sequence $\{1, 1, 0, 0, 1, 1, 1, \ldots\ldots \infty\}$.
8. Prove that these graphs G_1, G_3 and G_2, G_3 are non-isomorphic.



9. Find the domain of the following function :

$$f(x) = \sqrt{\log\left(\frac{5x - x^2}{4}\right)}$$

10. In how many ways can a team of 11 cricketers be chosen for 6 bowlers, 4 wicket keepers and 11 batsman to give majority of batsman so that at least 4 bowlers are there and 1 wicketkeeper?

PART - B

(Analytical/Problem solving questions). Attempt any five questions. **(5 × 8 = 40)**

1. (a) Write the scope and objective of DMS in Computer Science? **(4)**
   (b) In a test 70% of the candidates passed in Science, 65% in Mathematics, 27% failed in both Science and Mathematics and 124 passed in both the subjects. Find the total number of candidates for the test. **(4)**
2. Show that in the power set P(A) of all subsets of a set $A = \{a, b, c\}$, 'Set inclusion, $\subseteq$' is a partial order relation. Also draw the Hasse diagram for the POSET. **(6 + 2 = 8)**
3. (a) Solve the recurrence relations –

$$a_n - 5a_{n-1} + 6a_{n-2} = 3n^2 - 2n + 1$$ **(4)**

   (b) Prove by induction that sum of the cubes of three consecutive integers is divisible by 9. **(4)**
4. (a) Let $f : R \to R$ and $g : R \to R$ where R is the set of real numbers. Find gof and fog where $f(x) = x^2 - 2$ and $g(x) = x + 4$. State whether these functions are injective, surjective or bijective. **(4)**

(b) Draw the transition diagram of the finite state machine M (I, S, O, s_0, f, g), where I = {a, b}, S = {S_0, S_1}, O = {0, 1} and the transition table is as follows : **(4)**

S \ I	f		g	
	a	b	a	b
S_0	S_1	S_0	0	1
S_1	S_0	S_1	1	0

Also, find the output string for the input b b a a.

5. Define and explain the following by suitable examples : **(4 × 2 = 8)**

(i) Cyclic group (ii) Order of an element in a group (iii) Field (iv) Zero divisor of a ring

6. (a) Show that $\sim(p \vee (\sim p \wedge q)) \equiv (\sim p) \wedge (\sim q)$ (without truth table) **(4)**

(b) Write contrapositive converse and inverse of the statement "The home team wins whenever it is raining". Also construct the truth table for each statement. **(4)**

7. Write short notes on the following : **(4 × 2 = 8)**

(a) Planar graphs (b) Isomorphism of graphs (c) Cut sets (d) Vertex connectivity.

PART - C

(Descriptive/Analytical/Problem Solving/Design question). Attempt any four questions. **(4 × 15 = 60)**

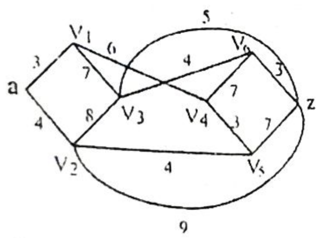
1. Let R = {(1, 2), (2, 3), (3, 1)} and A = {1, 2, 3}. Find reflexive, symmetric and transitive closure of R using : **(5 × 3 = 15)**

(a) Composition of relation R (b) Composition of matrix relation R (c) Graphical representation of R

2. (a) Define Bounded lattices, complement of an element of a lattices and distributive lattices. **(6)**

(b) Let (L, ≤) be a bounded distributive Lattice, if an element a ∈ L, has a complement then it is unique. **(9)**

3. (a) Find the shortest path from a to z in the following graph : **(5)**



(b) Suppose that a connected planar graph has 30 vertices, each of degree three. Into how many regions is the plane divided by a planar representation of this graph. **(5)**

(c) Let G be the set of all non-zero real numbers and Let $a * b = \frac{ab}{2}$, then show that (G_1 *) is an abelian group. **(5)**

4. (a) Obtain the Principal disjunctive normal forms of $(p \wedge q) \vee (\sim p \wedge r) \vee (q \wedge r)$. **(5)**

(b) Let Δ(G) be the maximum of the degrees of the vertices of a graph G then K(G) ≤ 1 + Δ(G) where K(G) is the chromatic number of graph. **(5)**

(c) In a complete graph with n – vertices there are $\frac{(n-1)}{2}$ edge disjoint Hamiltonian circuits, if n is an odd number ≥ 3.

5. (a) Define tautology and prove the following : **(5)**

$\{(p \to q) \wedge p\} \to q$ is tautology **(4)**

(b) Define fallacy and prove the following :

$(p \wedge q) \vee \sim (p \wedge q)$ is ∧ fallacy **(4)**

(c) Let (m, *) be a semi group and a ∈ m such that the equations a * u = x and v * a = x have solutions in M for all x ∈ M. Show that (M, *) is a monoid. **(7)**

□□□

It is predicted based and numeric value may change in exam so prepare the topics that i marked.

d number,
r function
ition to all

.T.U. 2017]

ion and a
he range
ed onto it
members

ets with
T.U. 2016]

n infinite
The best
f all real

finite set
et can be

ns with
T.U. 2016]

function
eger x is

quotient
provide
the mod
the div

For example, 23 mod 5 = 3, 18 mod 6 = 0, 23 div 5 = 4, and 5 div 6 = 0.

The mod function can determine the day of the week in n days from a given day.

Q.11 *Define the proof by contradiction with example.* *[R.T.U. 2016, 2013]*

Ans. Proof by contradiction : If in same case we have that

$\sim p \rightarrow q$, is true ... (A)

and also

$\sim p \rightarrow q$, is false ... (B)

But these are contradictory. As (A) and (B) are in contradiction one of them is false.

Ex. : Proove by contradictory that if

$x + y > 15$ then either $x > 10$ or $y > 5$

Sol. We assume the hypothesis $x + y > 15$. From here we must conclude that $x > 10$ or $y > 5$.

Assume to the contrary that

$x > 10$ or $y > 5$, is false

so $x \le 10$ and $y \le 5$

Adding both inequalities we get

$x + y \le 10 + 5 = 15$

which contradicts the hypothesis

$x + y > 15$

From here we conclude that the assumption "$x \le 10$ and $y \le 5$" cannot be true,

So "$x > 10$" or "$y > 5$" must be true.

Q.12 *Let f: R → R be a function defined as f(X) = 3X + 5 and g: R → R be another function defined as g (X) = X+4. Find (gof)$^{-1}$ and f^{-1}og^{-1} and verify (gof)$^{-1}$ = f^{-1}og^{-1}* *[R.T.U. 2015]*

Ans. Given that,

$f(x) = 3x + 5$, and so, $x = (f - 5)/3$, i.e., $f^{-1} = (x - 5)/3$

$g(x) = x + 4$, and so, $x = (g - 4)$, i.e., $g^{-1} = x - 4$

Now,

$gof = g(f(x))$

$= g(3x + 5)$

$= (3x + 5) + 4$

$= 3x + 9$

Now, $x = ((gof) - 9)/3$

So, $(gof)^{-1} = (x - 9)/3$

Now,

$f^{-1}og^{-1} = f^{-1}(g^{-1}(x))$

$= f^{-1}(x - 4)$

$= ((x - 4) - 5)/3$

$= (x - 9)/3$

Hence, we see that $(gof)^{-1} = f^{-1}og^{-1}$

Q.13 *Define the ceiling function with example* *[R*

Ans. Ceiling Functions : The closely-relate function, denoted by $\lceil x \rceil$ or ceil(x) or ceiling (function that returns the smallest integer nor less formally,

$$\lceil x \rceil = \min \{n \in Z : x \le n\}$$

For example, ceiling (2.3) = 3, ceiling (2) = 2 a (–2.3) = –2.

The names "floor" and "ceiling" and the corre notations were introduced by Kenneth E. Iverson i

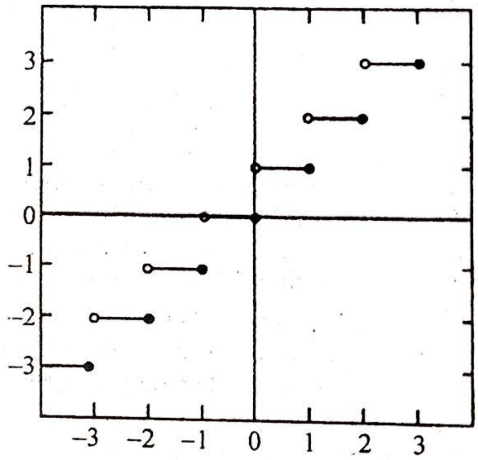


Fig. : The ceiling function

Q.14 *Define the remainder function with exam* *[R*

Ans. Remainder Function : A remainder function denoted by 'mod' or symbol '%') is one which numbers say, a,b returns the remainder when a is d b, i.e. "a mod b" is the remainder when a is divided

5 mod 2 = 1 (remainder when 5 is divided by

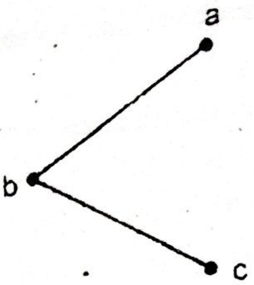
11 mod 3 = 2 (remainder when 11 is divided

Q.15 *Define the Reflexive relation.* *[I*

Ans. Reflexive Relation : A relation R on a non- is known as reflexive relation if each member related to itself, i.e. $x R x$ or $(x, x) \in R$, $\forall x \in A$

A relation R on a set A is irreflexive if $(x, x) \notin R$

Example 1. Let A be the set of all straight plane. The relation R in A defined by "x is parall reflexive, since every straight line is parallel to its



Hasse diagram

Q.25 *Write the scope and objective of DMS in Computer Science?* ***[R.T.U. 2019]***

Ans. Scope of D.M.S. : Though discrete mathematics has found application in almost every conceivable area of study. It is integral part of science course. It provides the mathematical foundation for many computer courses viz algorithms, database management, automata, compiler theory, operating system, computer language, to name a few with wrong mathematical foundation, these computer science subject become easy to understand.

Objectives : The objectives of this course is to provide the fundamental and concepts of Discrete Mathematical Structures with application of computer science including mathematical logic, Boolean Algebra, and its applications, switching circuits and logic gates. Groups and Trees, Important computer theorem with constructive proofs, real life problems and graphs, theoretic algorithms, to help the students to understand the computational and algorithmic assects of sets, relations, functions and algebric structure in field of computer science and its application.

Q.28 ***Let $A = Z$ the set of integers Relation R defined by A by aRb as 'a is congruent to b mod 2'. Show that R is an equivalence relations.*** **[R.T.U. 2013, 12]**

OR

Define congruency relation in Modulo system. If $A = Z$ (the set of integers), Relation R defined in A set by aRb as "a is congruent to b mod 2", then prove that R is an equivalence relation. **[R.T.U. 2017]**

OR

Let $A = Z$, the set of integers relation R define in A by aRb as "a congruent to b mod 2". Prove that R is an equivalence relation. **[R.T.U. 2014]**

Ans. Here R is $a \equiv b \pmod 2$ *i.e.* $a - b$ is divisible by 2

or $a - b$ is multiple of 2.

(i) Reflexive : Let $a \in A$, then

$a - a = 0 = 0 \times (2)$ a multiple of 2

$\Rightarrow aRa$

$\therefore$ R is reflexive.

(ii) Symmetric: Let $a, b \in A$, then

$aRb \Rightarrow a \equiv b \pmod 2 \Rightarrow a - b$ is divisible by 2

$\Rightarrow a - b = 2k, k \in Z$

$\Rightarrow b - a = -2k = 2(-k), -k \in Z$

$\Rightarrow b \equiv a \pmod 2$

$\Rightarrow bRa$

$\therefore$ R is symmetric.

(iii) Transitive: Let $a, b, c \in A$, then

$aRb \Rightarrow a - b = 2k_1;\ k_1 \in Z$ ....(i)

$bRc \Rightarrow b - c = 2k_2;\ k_2 \in Z$ ....(ii)

From eq. (i) + (ii)

$\Rightarrow a - c = 2(k_1 + k_2);\ k_1 + k_2 \in Z$

$\Rightarrow a \equiv c \pmod 2$

$\Rightarrow aRc$

$\therefore$ R is transitive.

Hence, it is an equivalence relation.

Now, tc
Let a ∈ A be ar
relation on A
(a, a) ∈ S. Hei
Now, S
then, (a, b) ∈
hence, (b, a)
that R ∩ S is
Suppos
Then, (a, b),
transitive, (a,
is transitive, (
(a, c) ∈ R an
R ∩ S is trans
Therefc

Q.30 ***An equi***
into equi
complete
such dis

Prove
decomp
either
set A i
classes.

Ans. By the de
[a] ⊆ A. We n
since A is refle
Hence A ⊆ U
establishes (i
[a] ∩ [b] = φ f
[a] ∩ [b] = φ.
means that c
R is symmetri
that the concl
The pro
xRa, since aR
that x ∈ [b]. T

Q.31 *(i) Prove, for finite sets A and B;*

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

(ii) In a class of 50 students, 15 play Tennis, 20 play Cricket and 20 play Hockey, 3 play Tennis and Cricket, 6 play Cricket and Hockey, and 5 play Tennis and Hockey, 7 play no game at all. How many play Cricket, Tennis and Hockey? *[R.T.U. 2014]*

Ans.(i) We know that

$$(A - B) \cup (A \cap B) \cup (B - A) = A \cup B \quad ...(1)$$

and $A - B$, $A \cap B$ and $B - A$ are pair wise disjoint therefore,

$$n(A \cup B) = n(A - B) + n(A \cap B) + n(B - A) \quad ...(2)$$

Further $A = (A - B) \cup (A \cap B)$

and $(A - B) \cap (A \cap B) = \phi$

so $n(A) = n(A - B) + n(A \cap B) \quad ...(3)$

Similarly

$$n(B) = n(A \cap B) + n(B - A) \quad ...(4)$$

Adding (3) and (4), we have

$$n(A) + n(B) = \{n(A - B) + n(A \cap B) + n(B - A)\} + n(A \cap B)$$
$$= n(A \cup B) + n(A \cap B) \text{[Using (2)]}$$

Thus

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

Ans.(ii) Let A = Student play Tennis

B = Student play Cricket

C = Student play Hockey

so $n(A) = 15$; $n(B) = 20$; $n(c) = 20$;

$N(A \cap B) = 3$; $n(B \cap C) = 6$; $n(A \cap C) = 5$

$\because$ 50 students in the class in which 7 play no game at all so

$$n(A \cup B \cup C) = 50 - 7 = 43$$

Now number of students those play Cricket, Hockey and Tennis is

$$N(A \cap B \cap C) = n(A \cup B \cup C) - n(A) - n(B) - n(C) + n(A \cap B) + n(B \cap C) + n(C \cap A)$$
$$= 43 - 15 - 20 - 20 + 3 + 6 + 5$$
$$= 57 - 55$$
$$= 2$$

Q.32 *(i) If f: A → B be one-one onto then the inverse map of f is unique. Prove it.*

(ii) Show that set of even positive integers is a countable set. *[R.T.U. 2014]*

Ans.(i) If possible, let g an

Then by the definitio

$g.f = f.g$

and $h.f = f.h$

where I is an identity

$h = h.I$

$= (h.f$

$= I.g$

$= g$

So $h = g$

Hence the result.

Ans. (ii) Proof : Let's coun

positive integers. We do tha

with the counting numbers:

1 2 3 4 5 6 7 8

2 4 6 8 10 12 14 1

This correspondence g

no even number that does

number. We can only deduc

number of elements. This n

is straightforward. The set

same number of elements

They are both infinite sets

integers (positive and nega

So does the set of all inte

all **countable sets,** because

elements as the counting nu

Q.33 *Compute the number elements.*

Ans. Here $n = 4$

Thus the number of p

$= S(4, 1) + S(4$

Now $S(4, 1) = 1 =$

$S(4, 2) = S(3,$

$S(4, 3) = S(3,$

Since $S(3, 2) = S(2,$

Now $S(2, 1) = S(2,$

Thus eq.(5) gives

$S(3, 2) = 1 + 2$

From (3)

$S(4, 2) = 1 + 2$

From (4)

$S(4, 3) = 3 + 3$

To find transitive arrow, we add arrow 1 to 1 since $1 \to 2 \to 3 \to 1$. Similarly, 2 to 2 and 3 to 3. Again we add arrow 1 to 3, since $1 \to 2 \to 3$. Similarly, 2 to 1 and 3 to 2. This is shown in Fig. 4.

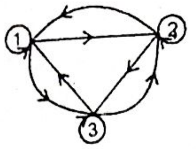


Fig. 4

Q.42 ***What is the Pigeonhole principle and the Extended Pigeonhole principle also prove both.*** *[R.T.U. 2012]*

OR

State and prove the generalized pigeonhole principle. *[R.T.U. 2017]*

OR

Explain pigeonhole and extended pigeonhole principle with example. *[R.T.U. 2010]*

OR

State and prove the Pigeonhole and Generalized Pigeonhole Principles. *[R.T.U. 2014]*

Ans. Pigeonhole Principle

If n pigeonholes are occupied by n pigeons and $m > n$ then at least one pigeonhole is occupied by more than one pigeon.

The pigeon hole principle is nothing more than the obvious remark : if you have fewer pigeon holes than pigeons and you put every pigeon in a pigeon hole, then there must result at least one pigeon hole with more than one pigeon. It is surprising how useful this can be as a proof strategy.

The pigeonhole principle, also known as *Dirichlet's* box (or drawer) principle, states that, given two natural numbers n and m with $n > m$, if n items are put into m pigeonholes, then at least one pigeonhole must contain more than one item. Another way of stating this would be that m holes can hold at most m objects with one object to a hole; adding another object will force one to reuse one of the holes, provided that m is finite. More formally, the theorem states that there does not exist an injective function on finite sets whose co-domain is smaller than its domain.

The pigeonhole principle is an example of a **counting argument** which can be applied to many formal problems, including ones involving infinite sets that cannot be put into one-to-one correspondence.

Generalizations of the Pigeonhole Principle

A generalized version of this principle states that, if n discrete objects are to be allocated to m containers, then at least one container must hold no fewer than $[n/m]$ objects,

where $[x]$ is the ceiling function, denoting the smallest integer larger than or equal to x.

A probabilistic generalization of the pigeonhole principle states that if n pigeons are randomly put into m pigeonholes with uniform probability 1/m, then at least one pigeonhole will hold more than one pigeon with probability

$$1-\frac{m!}{(m-n)!m^n}=1-\frac{(m)_n}{m^n}\quad\left(\because (m)_n=\frac{\sqrt{m+1}}{\sqrt{m-n+1}}\right)$$

where $(m)_n$ is falling factorial, for $n = 0$ and for $n = 1$ (and $m > 0$), that probability is zero; in other words, if there is just one pigeon, there cannot be conflict. For $n > m$ (more pigeons than pigeonholes) it is one, in which case it coincides with the ordinary pigeonhole principle. But even if the number of pigeons does not exceed the number of pigeonholes ($n \le m$), due to the random nature of the assignment of pigeons to pigeonholes there is often a substantial chance that clashes will occur. For example, if 2 pigeons are randomly assigned to 4 pigeonholes, there is a 25% chance that at least one pigeonhole will hold more than one pigeon; for 5 pigeons and 10 holes, that probability is 69.76%; and for 10 pigeons and 20 holes it is about 93.45%. This problem is treated at much greater length at birthday paradox.

Pigeonhole Principle : Simple Form

Theorem : If $n + 1$ objects are put into n boxes, then at least one box contains two or more objects.

Proof : Suppose none of the n boxes contains more than one object. Then the total number of objects would be at most n. This is a contradiction, since there are at least $n + 1$ objects.

Example : There are n married couples. How many of the 2n people must be selected in order to guarantee that one has selected a married couple?

Q.43 ***Prove by mathematical induction that $3^n > n^3$ for all integers $n \ge 4$.*** *[R.T.U. 2017]*

Ans. Let $P(n) = 3^n > n^3$ where $n \ge 4$.

Basic Step : P(n) is true since $3^4 = 81 > 4^3 = 64$

Inductive step : Let P(k) is true for all $k \ge 4$, i.e. $3^k > k^3$.

Then we need to show that P(k+1) is true, i.e. $3^{k+1} > (k+1)^3$

Let us rewrite

$$(k+1)^3 = k^3 + 3k^2 + 3k + 1$$

$$= k^3\left(1+\frac{3}{k}+\frac{3}{k^2}+\frac{1}{k^3}\right)$$

Since $3^k > k^3$ (using P(k)), we would be done if we could also prove that $3 > \left(1+\frac{3}{k}+\frac{3}{k^2}+\frac{1}{k^3}\right)$ for $k \ge 4$.

Observe that the function $f(k) = 1+\frac{3}{k}+\frac{3}{k^2}+\frac{1}{k^3}$

decreases as k increases, so that f(k) is largest when k is smallest. In other words, f(4) is the largest value of f(k), where $k \geq 4$.

Since

$$f(4) = 1+\frac{3}{4}+\frac{3}{4^2}+\frac{1}{4^3}$$

$$=\frac{125}{64}$$

is obviously less then 3. We have, for any integer $k \geq 4$,

$$3>\left(1+\frac{3}{k}+\frac{3}{k^2}+\frac{1}{k^3}\right)$$

Thus combining the two facts :

$3>\left(1+\dfrac{3}{k}+\dfrac{3}{k^2}+\dfrac{1}{k^3}\right)$ and $3^k > k^3$ for $k \geq 4$, we can multiply and get

$$3^{k+1} > k^3\left(1+\frac{3}{k}+\frac{3}{k^2}+\frac{1}{k^3}\right)$$

or $\quad 3^{k+1} > (k+1)^3$

So P(k+1) is true and then by mathematical induction P(n) is true for all integers $n \geq 4$, i.e. $3^n > n^3$.

Q.44 *Prove by mathematical induction that $6^{n+2} + 7^{2n+1}$ is divisible by 43 for each positive integer n.* *[R.T.U. 2016]*

OR

Define principle of mathematical induction and hence prove that $6^{(n+2)}+7^{(2n+1)}$ is divisible by 43 for each positive integer n. *[R.T.U. 2008]*

Ans. Principle of Mathematical Induction : Let p(n) be some statement involving a positive integer *n*. Suppose it is required to show that p(n) is true for integer *n* greater than some fixed integer n_0. The method adopted here to prove is called the *method of mathematical induction*. This method consists of the following three steps :

Basis Step : First it is verified whether p(n) is true for certain number. Generally we take n_0 to have the value one. If p (n) is not true for n = 1 then the least value of *n* found for which it is true.

Hypothesis : Then it is assumed that p (n) is true for n = k.

Inductive Step : Taking p(n) to be true for n = k, it is proved that p(n) is true also for the next value n = (k+1).

Since it has been found to be true for $n = n_0$, so it is true for $n = n_0 + 1$. When it is true for $n = n_0 + 1$, it is true for the next value $n = n_0 + 1 + 1$.

Arranging in this way, it is concluded that p(n) is true for all positive integer values of $n \geq n_0$. The above method of

proving a proposition p(n) involving a positive integer *n* is called the *Method of Mathematical Induction* or the *Principle of Mathematical Induction.*

Proof

Let $\quad p(n): 43$ divides $6^{n+2}+7^{2n+1}$

Basis Step : Let n = 1, then p(1) is true

$$= 6^{1+2}+7^{2+1} = 6^3+7^3 = 216+343$$

$= 559$, which is divisible by 43.

Inductive Hypothesis : Let p(k) be true for $k \geq 1$.

$6^{k+2}+7^{2k+1}$ is divisible by 43 (say 43λ).

Inductive Step : We shall show that

$p(k+1)$: $6^{k+3}+7^{2k+3}$ is divisible by 43, is true whenever p(k) is true.

$$6^{k+3}+7^{2k+3} = 6^{k+2}.\,6^1+7^{2k+1}.\,7^2$$

$$= 6^{k+2}.\,6+7^{2k+1}.(43+6)$$

$$= 43.\,7^{2k+1}+6\left(6^{k+2}+7^{2k+1}\right)$$

$$= 43.7^{2k+1}+6.43\lambda = 43\,(7^{2k+1}+6\lambda)$$

$\Rightarrow \quad 6^{k+3}+7^{2k+3}$ is divisible by 43.

Now

$\Rightarrow \quad$ p (k + 1) is true.

Hence by principle of mathematical induction 43 divides $6^{(n+2)}+7^{(2n+1)}$.

Q.45 *(a) Prove that $A-B = A \cap B' = B' \cap A'$*

(b) Consider the following collection of subsets $\{A_1, A_2, A_3\}$ of a set A = { 1,2,3,4,5,6,7,8,9,10}

[a] [{1,6,9}, {2,3,8}, {4,5,7,10}]

[b] {1}, {2,4,8}, {5,7,9}] and

[c] [{1,5}, {2,3,8}, {4,5,6,7,9,10}]

Determine which one is a partition of a set A

(c) Let f, g, h be mapping from N to N when N is the set of naturals such that f(n) = n + 1,

$$g(n) = 2n,\ h(n) = \begin{cases} 0, & n \text{ is even} \\ 1, & n \text{ is odd} \end{cases}$$

(i) Show that f, g and h are functions

(ii) Determine fof, fog, hog and (fog) oh

Where 'o' stands for composition of functions *[R.T.U. 2016]*

.....(1)
....(2)
....(3)

null set

= ϕ and

= ϕ

B = ϕ

B) = ϕ

PREVIOUS YEARS QUESTIONS

PART-A

Q.1 ***Define finite state machines.*** *[R.T.U. 2019]*

Ans. Finite automation as a machine equipped with an input tape. The machine works on a discrete time scale. At every point of time the machine is in one of its states, then it reads the next letter on the tape (the letter under the reading head), or maybe nothing (in the first variation), and then, according to the transition function (depending on the actual state and the letter being read, if any) it goes to a/the next state. It may happen in some variation that there is no transitions defined for the actual state and letter, then the machine gets stuck and cannot continue its run.

Q.2 ***In how many ways can a team of 11 cricketers be chosen for 6 bowlers, 4 wicket keepers and 11 batsmen to give majority of batsmen so that at least 4 bowlers are there and 1 wicketkeeper?*** *[R.T.U. 2019]*

Ans. 1 wicketkeeper can be selected in C(4, 1) ways

4 bowlers chosen = C(6, 4)

Remaining 6 batsmen = C(11, 6)

Total possibilities = C(4, 1) * C(6, 4) * C(11, 6) = 27720

The batsmen has to be majority. So the split cannot be 1 WC, 5 Bowlers, 5 Batsmen. It can only be 1WC, 4 bowlers and 6 batsmen.

Q.3 ***Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.*** *[R.T.U. 2017]*

Ans. $(p \wedge q) \rightarrow (p \vee q)$

First, we construct the truth table

p	q	$p \wedge q$	$p \vee q$	$p \wedge$
T	T	T	T	
T	F	F	T	
F	T	F	T	
F	F	F	F	

Since in the last column, all are tr
$p \wedge q \rightarrow p \vee q$ is a tautology.

Q.4 ***Find PCNF of a statement S w***
$(p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee$

Ans. First we obtain PDNF of ~s, (disjunction) of those minterms which given PDNF of s. Hence the PDNF of

$(\sim P \wedge \sim q \wedge \sim r) \vee (\sim P \wedge \sim q \wedge r)$

Thus, the PCNF of S ≡ [PDNF

$\equiv \sim ((\sim P \wedge \sim q \wedge \sim r) \vee (\sim P \wedge \sim q$

$\vee (P \wedge \sim q \wedge \sim r)$

$\equiv (P \vee q \vee r) \wedge (p \vee q \vee \sim r) \wedge (\sim$

Q.5 ***Explain the following for propos***

(i) Logical Equivalence

(ii) Tautological Implication

(iii) Normal Forms

Ans.(i) Logical Equivalence : Any which the truth table is same are sai EQUIVALENT.

Ex. $p \rightarrow q$ & $\sim p \vee q$

p	q	$p \to q$	$\neg q$	$(\neg p \vee q)$
F	F	T	T	T
F	T	T	T	T
T	F	F	F	F
T	T	T	F	T

(ii) Tautological Implication : Compound statements which are always true regardless of the truth or false of component statements are called tautologies. Obviously, the truth table of a tautology will contain only T entries in the last column.

Example : The statement $(p \to q) \leftrightarrow (\sim q \to \sim p)$ is a tautology

p	q	$p \to q$	$\sim q$	$\sim p$	$\sim q \to \sim p$	$(p \to q) \leftrightarrow (\sim q \to \sim p)$
T	T	T	F	F	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

(iii) Normal Forms : A literal L is either on after P or its negation $(\neg P)$. A clause D is a disjunction literals. A formula C is in NORMAL FORM if it is a conjunction of clause.

$L \to P | \neg P$
$D \to L | L \vee D$
$C \to D | D \wedge C$

Example : (a) $(\neg q \vee p \vee r) \wedge (\neg p \vee r) \wedge q$
(a) $(p \vee r) \wedge (\neg p \vee r) \wedge (p \vee \neg r)$

Q.6 *Over the universe of animals, let*
P(x) : x is a whale ; Q(x) : x is a fish
R(x) : x lives in water.
Translate the following into English
$\exists x (\sim R(x))$
$\exists x (Q(x) \wedge \sim P(x))$
$\forall x (P(x) \wedge R(x)) \to Q(x)$ *[R.T.U. 2014]*

Ans. $\exists x(\sim R(x))$: There exists an animal which does not lives in water.

$\exists x(Q(x) \wedge \sim P(x))$: There exists a fish that is not a whale.

$\exists x(P(x) \wedge R(x)) \to Q(x)$: Every whale that lives in the water, is a fish.

Q.7 *Consider the following :*
p : It is hot today
q : The temperature is 35°C
Write in simple sentence the meaning of the following :
(i) $p \vee q$ *(ii)* $\sim(p \vee q)$ *(iii)* $\sim(p \wedge q)$
(iv) $\sim p \wedge \sim q$ *[R.T.U. 2013]*

Ans. Given
p : It is hot today
q : The temperature is 35°C
(i) $p \vee q$
Either it is hot today or the temperature is 35° C
(ii) $\sim(p \vee q)$
Neither it is hot today nor the temperature is 35° C
(iii) $\sim(p \wedge q)$
It is not hot today and the temperature is not 35° C
(iv) $\sim p \vee \sim q$
It is not hot today and the temperature is not 35° C

PART-B

Q.8 **(a)** *Show that* $\sim(p \vee (\sim p \wedge q)) \equiv (\sim p) \wedge (\sim q)$ *(without truth table)*
(b) *Write contrapositive converse and inverse of the statement "The home team wins whenever it is raining". Also construct the truth table for each statement.* *[R.T.U. 2019]*

Ans.(a) $\sim(p \vee (\sim p \wedge q)) \equiv \sim p \wedge \sim(\sim p \wedge \sim q)$ [De Morgan's Law]
$\equiv \sim p \wedge (\sim\sim p \vee \sim q)$ [De Morgan's Law]
$\equiv \sim p \wedge (p \vee \sim q)$ [Double Negation Law]
$\equiv (\sim p \wedge p) \vee (\sim p \wedge \sim q)$ [Distributive Law]
$\equiv F \vee (\sim p \wedge \sim q)$ [$\therefore \sim p \wedge p \equiv F$]
$\equiv \sim p \wedge \sim p$

Ans.(b) The given proposition is in the form "q whenever p" such that,

q (Conclusion) : The home team wins.
p (hypothesis) : It is raining.

Converse : $q \to p$ is "If the home team wins then it is raining",

Inverse : $\neg p \to \neg q$ is "If it is not raining then the home team does not win".

Contra positive: $\neg q \to \neg p$ is "If the home team does not win then it is not raining".

p	q	$p \to q$	$q \to p$	$\sim p \to \sim q$	$\sim q \to \sim p$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

re is 35°C

lay or the temperature is 35° C

day nor the temperature is 35° C

and the temperature is not 35° C

nd the temperature is not 35° C

PART-B

(pv (~p ∧ q)) ≡(~p) ∧(~q) (without

apositive converse and inverse of it "The home team wins whenever '. Also construct the truth table for ent. *[R.T.U. 2019]*

$\equiv \sim p \wedge \sim(\sim p \wedge \sim q)$

[De Morgan's Law]

[De Morgan's Law]

[Double Negation Law]

~q) [Distributive Law]

$[\therefore \sim p \wedge p \equiv F]$

ition is in the form "q whenever p"

home team wins.

aining.

"If the home team wins then it is

s "If it is not raining then the home

→ ¬p is "If the home team does

g".

q → p	~p → ~q	~q → ~p

Q.9 ***Draw the transition diagram of the finite state machine M (I, S, O, S_0, f, g), where I = {a, b}, S = {S_0, S_1}, O = {0, 1} and the transition table is as follows-***

S \ I	f: a	f: b	g: a	g: b
S_0	S_1	S_0	0	1
S_1	S_0	S_1	1	0

Also, find the output string for the input b b a a.

[R.T.U. 2019]

Ans. FSM, M (I, S, O, S_0, f, g)

I = {a, b}

S = {S_0, S_1}

O = {0, 1}

A → I

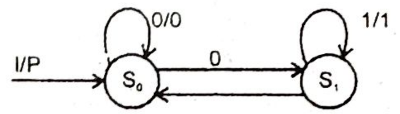
z → O

1. A finite set I of alphabet
2. A finite set S of internal state
3. A finite state Z of O/P symbol
4. An initial state S_0 in S
5. A next state function f from S × I into S.
6. An op state functions of from S × I into Z.

Transition of function $f(f : S \times I \Rightarrow P(S))$

S_0 - initial state

F - finite state



Transitional diagram (Result)

Output string for bbaa is 1101.

Q.10 ***In a test 70% of the candidate passed in Science, 65% in Mathematics, 27% failed in both Science and Mathematics and 124 passed in both the***

Ans. 70% passed in scie

27% failed in both.

100 − 27 = 73%

n(S) = 70%, n(M) =

n(S∪M) = n(S) + n(

= 70 + 65 − 73 = 62

∵ 62 passed in both su

∵ 124 passed in both s

$\therefore \frac{100}{62} \times 124 = 200.$

Q.11 ***Obtain the Princ (p ∧ q) ∨ (~p ∧***

Ans.

p	q	r	p∧q≡a
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	F
F	T	T	F
F	T	F	F
F	F	T	F
F	F	F	F

PDNF of (p∧q) ∨

= ((p ∧ q) ∧ (r∨ ~

= (p ∧ q ∧ r) ∨ (p

∨ (~ p ∧ r

= (p ∧ q ∧ r) ∨ (p

(Deletion of iden

Q.12 ***(a) Define Taut Determine t***

(i) If John

(ii) Only if

(b) Determine All men are All kings a Therefore,

Q.20 ***Show that propositional formula $(p \wedge q) \wedge (r \wedge s) \Rightarrow P$ for any propositions p,q,r,s is a tautology.*** *[R.T.U. 2013, 07]*

Ans. Given $(p \wedge q) \wedge (r \wedge s) \Rightarrow P$

p	q	r	s	$(p\wedge q)$	$(r\wedge s)$	$(p\wedge q)\wedge(r\wedge s)$	$(p\wedge q)\wedge(r\wedge s)\rightarrow p$
T	T	T	T	T	T	T	T
T	T	T	F	T	F	F	T
T	T	F	T	T	F	F	T
T	F	T	T	F	T	F	T
F	T	T	T	F	T	F	T
T	T	F	F	T	F	F	T
T	F	F	T	F	F	F	T
F	F	T	F	F	F	F	T
F	T	T	F	F	F	F	T
T	F	T	T	F	T	F	T
F	T	F	F	F	F	F	T
T	F	F	F	F	F	F	T
F	F	F	T	F	F	F	T
F	F	T	F	F	F	F	T
F	T	F	F	F	F	F	T
F	F	F	F	F	F	F	T

$\because (p \wedge q) \wedge (r \wedge s) \rightarrow p$ is always true

Hence proved that the given propositional formula is tautology.

POSETS, HASSE DIAGRAM AND LATTICES

3

PREVIOUS YEARS QUESTIONS

PART-A

Q.1 ***Prove that α^2 is an even integer, then α is an even integer.*** ***[R.T.U. 2019]***

Ans. Let a α^2 is an even integer

$\alpha^2 = 2k$, for some integer k

$$\alpha = \frac{2k}{\alpha}$$

So there is an integer $J = \frac{k}{\alpha}$, such that $\alpha = 2J$ α is an even.

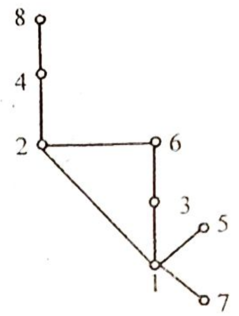
Q.2 ***Give an example of a partially ordered set which is not a lattice.*** ***[R.T.U. 2019]***

Ans. A partially ordered set $(X, \leq)$ is called a lattice if for every pair of elements x, y $\in$ X both the infimum and suprememum of the set {x, y} exists. I am trying to get an intuition for how a partially ordered set can fail to be a lattice. In R, for example, once two elements are selected the completeness of the real numbers guarantees the existence of both the infimum and supremum. Now, if we restrict our attention to a nondegenerate interval (a, b) it is clear that no two points in (a, b) have either a suprememum or infimum in (a, b).

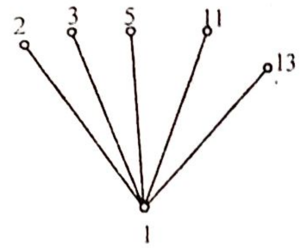
Q.3 ***Draw a Hasse Diagram for (A), (divisibility relation), where***
(i) A = {1,2,3,4,5,6,7,8}
(ii) A = {1,2,3,5,11,13};
(iii) A = {2,3,4,5,6,30,60};
(iv) S = {1,2,3,6,12,24,};

Ans.

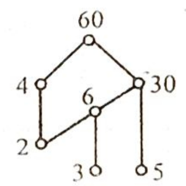
(i) A = {1,2,3,4,5,6,7,8}



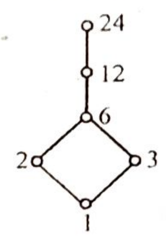
(ii) A = {1,2,3,5,11,13}



(iii) A ={2,3,4,5,6,30,60}



(iv) S = {1,2,3,6,12,24,}



Q.4 *Solve the following:*

(a) *In a city, the bus route numbers consists of natural number less than 100, followed by o[...] of the letters A,B,C,D,E and F. How ma[...] different bus routes are possible?*

(b) *There are 3 question in a question paper. the questions have 4,3 and 2 solutio[...] respectively, find the total number of solutio[...]*

Ans.(a) The number can be any me of the natural numbe[...] from 1 to 99.

There are 99 choices for the number.

The letter can be chooses in 6 ways

Number of possible bus routes are : $99 \times 6 = 594$

Ans.(b) Here question 1 has 4 solutions, question 2 has 3 solutions and question 3 has 2 solutions.

By the multiplication rule,

Total number of solutions: $4 \times 3 \times 2 = 24$

Q.5 ***Find the generating functions for the following sequences:***

(a) 1,1,1,1,1,1,0,0,0,0,....

(b) 1,3,3,1,0,0,0,0,....

Ans.(a) The generating function is:

$$G(x) = 1+1x+1x^2+1x^3+1x^5+0x^6+0x^7+\ldots$$
$$= 1+x+x^2+x^3+x^4+x^5$$

We can apply the formula for the sum of a geometric series to rewrite G(x) as

$$G(x) = \frac{1-x^6}{1-x}$$

Ans.(b) The generating function is

$$G(x) = 1+3x+3x^2+1$$

Using formula

$$G(x) = (1+x)^3$$

PART-B

Q.6 *(a) Solve the recurrence relations-*

$$a_n - 5a_{n-1} + 6a_{n-2} = 3n^2 - 2n + 1$$

(b) Prove by induction that sum of the cubes of three consecutive integers is divisible by 9.

[R.T.U. 2019]

Ans.(a) $a_n - 5a_{n-1} + 6a_{n-2} = 3n^2 - 2n + 1$

Particular solution of the above is if the form

$$a_n = P_1n^2 + P_2n + P_3$$

$P_1n^2 + P_2n + P_3 - 5P_1(n-1)^2 - 5P_2(n-1) - 5P_3 + 6P_1(n-2)^2 + 6P_2(n-2) + 6P_3 = 3n^2 - 2n + 1$

$\Rightarrow$ $P_1n^2 + P_2n + P_3 - 5P_1(n^2 - 2n + 1) - 5P_2n + 5P_2 - 5P_3 + 6P_1(n^2 - 4n + 4) + 6P_2n - 12P_2 + 6P_3 = 3n^2 - 2n + 1$

$\Rightarrow$ $P_1n^2 + P_2n + P_3 - 5P_1n^2 - 10P_1n - 5P_1 - 5P_2n + 5P_2 - 5P_3 + 6P_1n^2 - 24P_1n + 24P_1 + 6P_2n - 12P_2 + 6P_3$

$= 3n^2 - 2n + 1$

$\Rightarrow$ $n^2(P_1 - 5P_1 + 6P_1) + n(P_2 + 10P_1 - 5P_2 - 24P_1 + 6P_2) + (P_3 - 5P_1 + 5P_2 - 5P_3 + 24P_1 - 12P_1 + 6P_3)$

$= 3n^2 - 2n + 1$

Equating coefficients of n^2, n and constant term, we get

$$P_1 - 5P_1 + 6P_1 = 3$$
$$P_1 + P_1 = 3$$
$$2P_1 = 3$$
$$P_1 = \frac{3}{2}$$
$$P_2 + 10P_1 - 5P_2 - 24P_1 + 6P_2 = -2$$
$$P_2 + 10 \times \frac{3}{2} - 5P_2 - 24 \times \frac{3}{2} + 6P_2 = -2$$
$$P_2 + 15 - 5P_2 - 36 + 6P_2 = -2$$
$$2P_2 = -2 - 15 + 36$$
$$P_2 = \frac{19}{2}$$
$$P_3 - 5P_1 + 5P_2 - 5P_3 + 24P_1 - 24P_1 + 6P_3 = 1$$
$$19P_1 + 2P_3 - 7P_2 = 1$$
$$19 \times \frac{3}{2} + 2P_3 - 7 \times \frac{19}{2} = 1$$
$$19 \times 3 + 4P_3 - 7 \times 19 = 2$$
$$4P_3 = 2 + 7 \times 19 - 19 \times 3$$
$$P_3 = \frac{78}{4}$$

so the particular solution is

$$a_nP = \frac{3}{2}n^2 + \frac{19}{2}n + \frac{78}{4}$$

Ans. (b) $P(n) = m^3 + (m+1)^3 + (m+2)^3$ is divide by 9.

$P(n) = m^3 + (m+1)^3 + (m+2)^3 = a\lambda$

$P(1) = 1^3 + (1+1)^3 + (1+2)^3$

$= 1 + 8 + 27 = 36 = 9 \times 4$

aX* → P(1) is true

Let P(m) be true

$P(m) : m^3 + (m+1)^3 + (m+2)^3 = a\lambda$

G $P(m+1) : (m+1)^3 + (m+2)^3 + (m+3)^3 = aK$

$(m+1)^3 + (m+2)^3 + (m+3)^3 = (m+1)^3 + (m+2)^3 + m^3 + 9m^3 + 27m + 27$

$\Rightarrow$ $m^3 + (m+1)^3 + (m+2)^3 + 9m^3 + 27m + 27 = a\lambda + am^2 + 27m + 27$

$\Rightarrow$ $a(\lambda + m^2 + 3m + 3) = 9k$

$= \frac{P(1)}{P(m+1)}$ is true.

Q.7 ***If the coefficient of $(2r+4)^{th}$ and $(r-2)^{th}$ terms in the expansion of $(1+\chi)^{18}$ are equal, then find the value of r.***

Ans. The general term of $(1+x)^n$ is T_{r+1} C_rx^r

Hence coefficient of $(2r+4)^{th}$ term will be

$$T_{2r+4} = T_{2r+3+1} = {}^{18}C_{2r+3}$$

and coefficient or $(r-2)^{th}$ term will be

$$T_{r-2} = T_{r-3+1} = {}^{18}C_{r-3}$$
$$^{18}C_{2r+3} = {}^{18}C_{r-3}$$
$$(2r+3) + (r-3) = 18$$

$(\because {}^nC_r = {}^nC_k = r = k \text{ or } r + k = n)$

$\therefore$ $r = 6$

Q.8 ***In a lattice defined by the following Hasse Diagram, how many complements does the element 'e' have?***

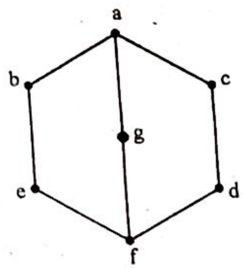


Fig.

Ans. The element e has 3 components – g,c and d.

$$e \vee g = a \text{ and } e \wedge g = f$$
$$e \vee c = a \text{ and } e \wedge c = f$$
$$e \vee d = a \text{ and } e \wedge d = f$$

Q.6 *Write short note on Galois field.* *[R.T.U 2012]*

Ans. Galois Field: A finite field is a field with a finite field order (i.e., number of elements), also called a Galois field. For example, GF(p) is called the prime field of order p, and is the field of residue classes modulo p, where the p elements are denoted 0, 1, ..., $p-1$. $a = b$ in GF(p) means the same as $a \equiv b$. (mod p)

The finite field GF(2) consists of elements 0 and 1 which satisfy the following addition and multiplication tables.

+	0	1
0	0	1
1	1	0

×	0	1
0	0	0
1	0	1

Q.7 *Explain Subgroup.*

Ans. Let G be a group and 4 $\subseteq$ G be non-empty. If H is also a group under the same operation as G, then H is a sub group of G. If $\{e\} \subseteq H \subseteq G$ then H is proper subgroup of G.

PART-B

Q.8 *Define and explain the following by suitable examples-*
(i) Cyclic group
(ii) Order of an element in a group
(iii) Field
(iv) Zero divisor of a ring *[R.T.U. 20]*

Ans.(i) Cyclic Group : In group theory, a branch of abstract algebra, a cyclic group or monogenous group is a group that is generated by a single element. That is, it is a set of invertible elements with a single associative binary operation, and it contains an element g such that every other element of the group may be obtained by repeatedly applying the group operation to g or its inverse. Each element can be written as a power of g in multiplicative notation, or as a multiple of g in additive notation. This element g is called a generator of the group.

Example:

Integer and modular addition

The set of integers Z, with the operation of addition, forms a group. It is an infinite cyclic group, because all integers can be written by repeatedly adding or subtracting the single number 1. In this group, 1 and "1 are the only generators. Every infinite cyclic group is isomorphic to Z.

For every positive integer n, the set of integers modulo n, again with the operation of addition, forms a finite cyclic group, denoted Z/nZ. A modular integer i is a generator of this group if i is relatively prime to n, because these elements can generate all other elements of the group through integer addition. (The number of such generators is $\varphi(n)$, where φ is the Euler totient function.) Every finite cyclic group G is isomorphic to Z/nZ, where n = |G| is the order of the group.

The addition operations on integers and modular integers, used to define the cyclic groups, are the addition operations of commutative rings, also denoted Z and Z/nZ or Z/(n). If p is a prime, then Z/pZ is a finite field, and is usually denoted F_p or GF(p).

(ii) Order of a group : In group theory, a branch of mathematics, the order of a group is its cardinality, that is, the number of elements in its set. The order of an element a of a group, sometimes also period length or period of a, is the smallest positive integer m such that $a^m = e$, where e denotes the identity element of the group, and a^m denotes the product of m copies of a. If no such m exists, a is said to have infinite order.

The order of a group G is denoted by ord(G) or |G|, and the order of an element a is denoted by ord(a) or |a|. The order of an element a is equal to the order of its cyclic subgroup <a> = {a^k for k an integer}, the subgroup generated by a. Thus, |a| = |<a>|.

Lagrange's theorem states that for any subgroup H of G the order of the subgroup divides the order of the group: |H| is a divisor of |G|. In particular, the order |a| of any element is a divisor of |G|.

Example. The symmetric group S_3 has the following multiplication table.

•	e	s	t	u	v	w
e	e	s	t	u	v	w
s	s	e	v	w	t	u
t	t	u	e	s	w	v
u	u	t	w	v	e	s
v	v	w	s	e	u	t
w	w	v	u	t	s	e

This group has six elements, so ord(S_3) = 6. By definition, the order of the identity, e, is one, since e^1 = e. Each of s, t, and w squares to e, so these group elements have order two: |s| = |t| = |w| = 2. Finally, u and v have order 3, since u^3 = vu = e, and v^3 = uv = e.

(iii) Field : *Refer to Q.4.*

Example :

Rational Numbers : Rational numbers have been widely used a long time before the elaboration of the concept of field. They are numbers that can be written as fractions a/b, Where a and b are integers, and b ≠ 0. The additive inverse of such a fraction is –a/b, and the multiplicative inverse (provided that a ≠ 0) is b/a, which can be seen as follows :

$$\frac{b}{a}\cdot\frac{a}{b} = \frac{ba}{ab} = 1$$

The abstractly required field axioms reduce to standard properties of rational numbers. For example, the law of distributivity can be proven as follows :

$$\frac{a}{b}\cdot\left(\frac{c}{d}+\frac{e}{f}\right)$$

$$= \frac{a}{b}\cdot\left(\frac{c}{d}\cdot\frac{f}{f}+\frac{e}{f}\cdot\frac{d}{d}\right)$$

$$= \frac{a}{b}\cdot\left(\frac{cf}{df}+\frac{ed}{fd}\right) = \frac{a}{b}\cdot\frac{cf+ed}{df}$$

$$= \frac{a(cf+ed)}{bdf} = \frac{acf}{bdf}+\frac{aed}{bdf} = \frac{ac}{bd}+\frac{ae}{bf}$$

$$= \frac{a}{b}\cdot\frac{c}{d}+\frac{a}{b}\cdot\frac{e}{f}$$

(iv) Zero Divisors in Rings

Definition : Let (R,+, *) be a ring where 0 ∈ R is the identity of +. The element a ∈ R\{0} is said to be a Zero-Divisor of R if there exists a b ∈ R\{0} such that a * b = 0 or b * a = 0.

For example, consider the ring (M_{22}, +, *) of 2 × 2 matrices with real coefficients and with the operations of standard matrix addition +, and standard matrix multiplication *. Recall that the identity of + is the 2 × 2 zero matrix

$$0 = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

Further consider the matrices A, B ∈ M_{22} given by

$$A = \begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}, B = \begin{bmatrix}0 & 1\\ 0 & -1\end{bmatrix} \quad ...(1)$$

When we multiply the matrices A and B together we have that

$$A * B = \begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ 0 & -1\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} = 0 \quad ...(2)$$

Notice that A is not the identity for + and B is not the identity for +. Therefore, the matrices A and B are zero divisors of M_{22}.

We should bee clear that a ring (R, +, *) need not have any zero divisors. For example, consider the ring (C, +, *) of complex numbers where + is standard addition and * is standard multiplication. We note that the identity of + is 0 = 0 + 0i ∈ C. Since (C, +, *) is a commutative ring, then for x, y ∈ C\{0} and where x = a + bi and y = c + di for a, b, c, d ∈ R we need to consider the following equation.

$$x * y = (a+bi)(c+di) = (ac-bd)+(ad+bc)i = 0+0i \quad ...(3)$$

Note that this equality holds if and only if :

$$ac = bd \text{ and } ad + bc = 0 \quad ...(4)$$

Without loss of generality, assume x ≠ 0. Then either a ≠ 0 or b ≠ 0 or both. Assume a ≠ 0. Then we can divide both equations by a to get :

$$c = \frac{bd}{a}(*) \text{ and } d+\frac{bc}{a} = 0(**) \quad ...(5)$$

Substituting the first equation into the second yields:

$$d + c\frac{b}{a} = 0(*) \quad ...(6)$$

$$d + \frac{bd}{a}\cdot\frac{b}{a} = 0$$

$$d + \frac{b^2 d}{a^2} = 0(*)$$

$$d\left(1+\frac{b^2}{a^2}\right) = 0$$

There are two possibilities in the equation above. Either $d = 0$ or $1+\frac{b^2}{a^2} = 0$. Clearly $1+\frac{b^2}{a^2} = 0$ since this would imply that $\left(\frac{b}{a}\right)^2 = -1$. Therefore d = 0.

Looking at (**) we see that then $\frac{bc}{a} = 0$ so either b = 0 or c = 0. If c = 0 we have that then y = c + di = 0. Meanwhile, if c ≠ 0 then b = 0 and by (*) this implies that c = 0 so then y = c + di = 0 again. In either case, we see that if x = a + bi ≠ 0 + 0i then y = c + di = 0 + 0i. Therefore, there exists no zero divisors in the ring (c, +, *)

$$\text{LHS} = (a, b) \odot (c + e, d + f)$$

$$= (ac + ae - bd - bf, bc + be + ad + af)$$

$$\text{RHS} = ((a,b) \odot (c, d)) \oplus ((a, b) \odot (e, f))$$

$$= (ac - bd, bc + ad) \oplus (ae - bf, be + af)$$

$$= (ac - bd + ae - bf, bc + ad + be + af)$$

$$\text{LHS} = \text{RHS}$$

Hence Proved

Q.11 *The necessary and sufficient condition for a non-empty subset H of a group {G, *} to be a subgroup is a, b ∈ H ⇒ $a*b^{-1} \in H$.* **[R.T.U. 2016]**

Ans. The condition is necessary. Suppose H is a subgroup of G and let a ∈ H, b ∈ H. Now each element of H must posses inverse because H itself is a group.

$$b \in H \Rightarrow b^{-1} \in H$$

Also, H is closed under composition * in G. Therefore,

$$a \in H, b^{-1} \in H \Rightarrow a * b^{-1} \in H$$

The condition is sufficient. If it is given $a \in H, b^{-1} \in H \Rightarrow a * b^{-1} \in H$, then we have to prove that H is a subgroup.

(i) **Closure property :** Let a, b ∈ H then $b \in H \Rightarrow b^{-1} \in H$.

Therefore, $a \in H, b^{-1} \in H \Rightarrow a * (b^{-1})^{-1} \in H$.

$$\Rightarrow a * b \in H.$$

H is closed with respect to composition * in G.

(ii) **Associative property :** Since elements of H are also the elements of G, the composition is associative in H.

(iii) **Existence of identity :** Since,

$$a \in H, a^{-1} \in H \Rightarrow a * a^{-1} \in J = e \in H.$$

(iv) **Existence of inverse :** Let a ∈ H then

$$e \in H, a \in H \Rightarrow e * a^{-1} \in H \Rightarrow a^{-1} \in H.$$

Hence, H itself is a group for the composition * in group G.

Q.12 *Show that $z_5 = \{0,1,2,3,4\}$ is an abelian group for the operation $+_5$ defined as.*

$$a +_5 b = \begin{cases} a+b & \text{if } a+b<5 \\ a+b-5 & \text{if } a+b \geq 5 \end{cases}$$

[R.T.U. 2015]

$\Rightarrow e \in H$

Thus the identity e is an element of H.

Existence of inverse: Let a be any element of H. Then by the given condition we have

$e \in H,\ a \in H \Rightarrow e\ a^{-1} \in H \Rightarrow a^{-1} \in H$

Thus each element of H possess inverse.

Closure property : Let a, b ∈ H

$\because\ b \in H \Rightarrow b^{-1} \in H$

Therefore applying the given condition, we have

$a \in H,\ b^{-1} \in H \Rightarrow a(b^{-1})^{-1} \in H \Rightarrow ab \in H$

Associativity : The elements of H are also the elements of G. The composition in G is associative. Therefore it must also be associative in H.

Hence H itself is a group for the composition in G. Therefore H is a subgroup of G.

PART-C

Q.14 (a) Let (m, *) be a semi group and a ∈ m such that the equations a * u = x and v * a = x have solutions in M for all x ∈ M. Show that (M, *) is a monoid.

(b) Let Δ(G) be the maximum of the degrees of the vertices of a graph G then K(G) ≤ 1 + Δ(G) where K(G) is the chromatic number of graph.

(c) Let G be the set of all non-zero real numbers and Let $a * b = \frac{ab}{2}$, then show that (G, *) is an abelian group.

[R.T.U. 201]

Ans.(a) Given a * u = x

and v * a = x; ∀x ∈ A ...(

a ∈ A if we take x = a equation (i) are satisfied.

For some $u = e_l$ and $v = e_m$

$a * b\ e_l = a$ and $e_m * a = a$

Again Let y ∈ A

$y * e_l = (v * a) * e_l = v * (a * e_l) = v * a = y$

$e_m * y = e_m * (a * u) = (e_l * a) * u$

$= a * u = y$

$y * e_l = y$ and $e_m * y = y$

e_l and e_m are right and left identity in A

$e_l = e_m = e$

identity element e ∈ A.

Ans.(b) Proof : Let the no of vertex in a graph is denoted by |V|. If |V| = 1 then A(G) = 0, K(G) = 1. So the results holds. Now let k be an integer. and k ≥ 1. Assume that results hold for all graph with |V| = k vertex.

Let G be a graph with (k + 1) vertex

Let V be any vertex of G and $\widetilde{G_0} = G\{v\}^2$

is a subgraph of G obtained by deleting V from G.

Since G_0 has k vertex so we use induction

$k(G_0) \le 1 + \Delta(G_0)$

$\Delta(G_0) \le \Delta(G_0)$

$K(G_0) \le 1 + \Delta(G)$

So G_0 can be colored with atmost 1 + Δ(G) colors. Since there can be almost Δ(G) vertices adjacent to V, one of the available 1 + Δ(G) colors remains for V. Thus G can be colored with atmost 1 + Δ(G) colors.

Ans.(c) Let a, b ∈ G Here $a * \beta = \frac{ab}{2}, \forall a, b \in G$

(i) **Closure :** a and b are non zero real numbers ab is also a non zero real number

$\frac{ab}{2}$ is also a non zero real number

$\frac{ab}{2} \in G \Rightarrow (a * b) \in G$

G is closed

(ii) **Associativity :** Let e a, b, c ∈ G

Then

$$a * (b * c) = a * \left(\frac{bc}{2}\right) = \frac{a(bc)}{2.2} = \frac{abc}{4} \quad ...(1)$$

$$(a * b) * c = \left(\frac{ab}{2}\right) * c = \frac{a(bc)}{2.2} = \frac{abc}{4} \quad ...(2)$$

From (1) and (2)

a * (b * c) = (a * b) * c

G is associative

(iii) Identity : Let e be the identity element is G.

a * e = a ∀ a ∈ G.

$\frac{ae}{2} = a \Rightarrow e = 2 \in G$

2 is identity element is G.

(iv) Inverse : Let a' b c the inverse of a ∈ G.

a * a' = 2

$\frac{aa'}{2} = 2 \Rightarrow a' = \frac{4}{a} \in G(a \neq 0)$

a ∈ G, So each elements of G has its inverse in G.

(v) Abelian : Let a, b ∈ G then

$a * b = \frac{ab}{2} = \frac{ba}{2} = b * a$

$a * b = b * a.\ \forall a, b \in G$

Therefore (G, *) is an abelian.

Q.15 (a) Prove that every infinite cyclic group has two and only two generators.

(b) Show that the set $t(g) = \left\{\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} a \in z, b \in z\right\}$ is an ideal of the ring $R = \left\{\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} a, b, c \in z\right\}$.

Matrix addition and matrix multiplication being the operations of the system. *[R.T.U. 2014]*

Ans.(a) Let G = {a} be an infinite cyclic group generated by a. The elements of G will be integral power of a.

We claim that no two distinct integral power of a can be equal.

For, if possible, let $a^r = a^s$, r > s

$\Rightarrow\ a^r \cdot a^{-s} = a^{s-s}$

$\Rightarrow\ a^{r-s} = a^0$

$\Rightarrow\ a^{r-s} = 0$

Since r–s is positive integer

$a^{r-s} = 1 \Rightarrow a^0 = r - s$ finite

So 'a' can't be a generator of an infinite cyclic group G.

Hence $a^r \neq a^s$ unless r = s

Therefor we can write

$G = \{...a^{-4}, a^{-3}, a^{-2}, a^{-1}, a^0, a^1, a^2, ...\}$

If a^r is any elment of G we can write $a^r = (a^{-1})^{-r}$

Thus a^{-1} is also a generator of G. To show that a and a^{-1} generator Now if m ≠ 1 or –1 then a^m can't be generator of G. If a^m is to a generator of G, there must exist an integer k such that $(a^m)k = a$ i.e. $a^{mk} = a$

Now m = 1 or –1 Þ mk ≠ 1

Therefore two distinct integral powers of 'a' are equal and this contracts that statement we have just proved. Hence a^m cannot be a generator of G if m ≠ 1 or – 1. Thus G has exactly two generators.

Graph Theory 5

Previous Years Questions

Part-A

Q.1 *Write short note on isomorphism of graphs.* **[R.T.U. 2019]**

OR

Define the isomorphic graph with example. **[R.T.U. 2014]**

Ans. Isomorphic Graphs : Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to be Isomorphic to each other if there exists a bijection mapping f from V_1 to V_2.

i.e., $f : V_1 \to V_2$ such that for each of the vertices v_i, v_j of V_1,

$\{v_i, v_j\} \in E_1 \Rightarrow \{f(v_i), f(v_j)\} \in E_2$

The function f is called an Isomorphism from G_1 to G_2.

It is immediately apparent by the definition of isomorphism that two isomorphic graphs must have

(a) The same number of vertices

(b) The same number of edges

(c) An equal number of vertices with a given degree *i.e.*, same degree sequence.

However, these conditions are by no means sufficient. For instance, the two graphs (given below) satisfy all three conditions, yet they are not isomorphic.

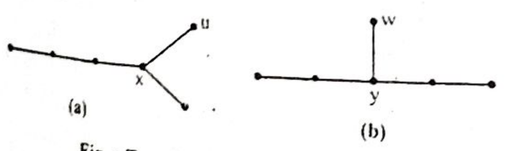


Fig. : Two graphs that are not isomorphic

The graphs in fig. (a) and (b) are not isomorphic can be shown as follows : If the graph (a) were to be isomorphic to the one in (b), vertex x must correspond to y, because there are no other vertices of degree three. Now in (b) there is only one pendant vertex, w, adjacent to y. While in (a) there are two pendant vertices, u and v, adjacent to x. Thus the adjacency relationship is not preserved. Hence (a) and (b) are not isomorphic.

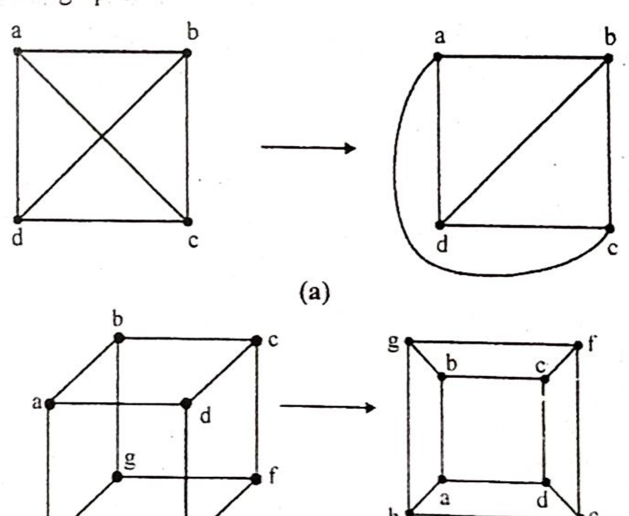
Q.2 *Write short note on planar graphs.* **[R.T.U. 2019]**

OR

Define the planar graph with example. **[R.T.U. 2014]**

Ans. Planar Graphs : A graph is called planar if it can be drawn in a plane such that no two edges intersect except at their common end vertices, if any.

Note that, if a graph G has been drawn with crossing edges, it does not mean that G is non–planar. There may be other planar representation of G. For example, following are planar graphs.



(a)

(b)

Fig.

Q.3 *Define the*

Ans. Weighted G
associated with
weights to be no
also known as e

Example: T
right.

Q.4 *Write sh*
Graphs.

Ans. Eulerian
passes through
at the first ver
Euler line in C
called an *Eul*

Hamilt
which has a cl
once though i

Q.5 *Write*

Ans. Cut S
edges with

- T
- T
d

As an

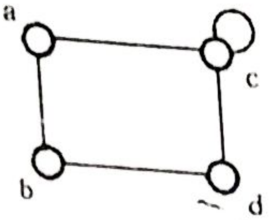


Fig.

The above G cannot be disconnected by removing a single vertex, but the removal of two non-adjacent vertices (such as b and c) disconnects it. The G has connectivity 2.

Q.7 ***Prove that the chromatic number of a graph will not exceed by more than one, the maximum degree of the vertices in a graph.*** *[R.T.U. 2017, 2011]*

Ans. Proof : Let $\Delta(G)$ be the maximum of the degrees of the vertices of a graph G.

Let the number of vertices in a graph is denoted by $|V|$. If $|V| = 1$, then $\Delta(G) = 0$ and $K(G) = 1$, so the result holds. Now let K be an integer and $K \geq 1$. Assume that the result holds for all graph with $|V| = K$ vertices.

Let G be a graph with $(K + 1)$ vertices. Let v be any vertex of G and let $G_0 = G/\{v\}$ is a subgraph of G obtained by deleting v from G.

Since G_0 has K vertices so we can use the induction hypothesis to conclude that

$$K(G_0) \leq 1 + \Delta(G_0)$$

Also, $\Delta(G_0) \leq \Delta(G)$

$\therefore \quad K(G_0) \leq 1 + \Delta(G)$

So G_0 can be colored with at most $1 + \Delta(G)$ colors. Since there can be atmost $\Delta(G)$ vertices adjacent to v, one of the available $1 + \Delta(G)$ colors remains for v. Thus G can be colored with atmost $1 + \Delta(G)$ colors. **Hence proved.**

Q.8 ***Show the total number of odd degree vertices of a (p, q), graph (graph with p vertices and q edges) is even.*** *[R.T.U. 2012]*

OR

Prove that the number of vertices of odd degrees in an undirected graph is always even. *[R.T.U. 2016]*

OR

Prove that the number of odd degree vertices in a graph G is always even. *[R.T.U. 2011, 2009; Raj. Univ. 2007, 2006, 2005]*

Ans. Let G (V, E) be a graph. $V_e \subset V$ and $V_o \subset V$ be the set of vertices of even degree and odd degree respectively. The $V_e \cup V_o$. Also let n be the number of edges.

$$\sum_{v \in V} \deg(v) = \sum_{v \in V_e} \deg(v) + \sum_{v \in V_o} \deg(v) = 2n$$

Since deg(v) is even for $v \in V_e$ $\sum_{v \in V_e} \deg(v) = m$ (say) is also even.

$$\Rightarrow \sum_{v \in V_o} \deg(v) = 2n - m = 2n - 2k$$

(m = 2k for some integer k)

$= 2(n - k) = 2l$ ($l = n - k$ is an integer)

$=$ an even number

Since all the terms in the sum $\sum_{v \in V_o} \deg(v)$ are odd, there must be an even number of such terms. Thus, number of odd degree vertices is even.

Q.9 ***(a) Give an example of connected graph that has***
(i) A Hamiltonian cycle but no Euler circuit
(ii) A Euler circuit but no Hamiltonian cycle
(b) What is the length of shortest path between the vertices a to z in the following weighted graph.

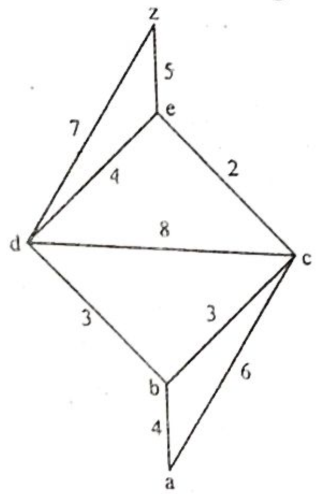


Fig.

[R.T.U. 2016]

Ans.(a)(i) A Hamiltonian cycle but no Euler circuit

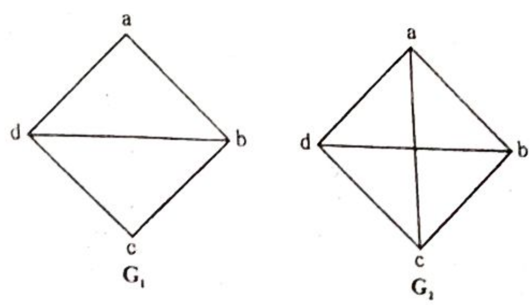
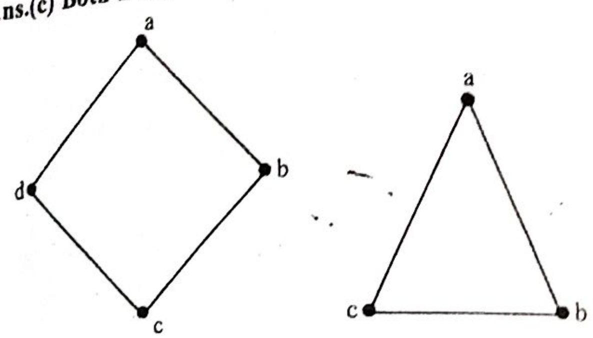


Fig. 1

Ans.(c) Both Euler circuit and Hamiltonian cycle :



G₁ G₂

Fig.

In G_1 Euler circuit : a, b, c, d, a

Hamiltonian cycle : a, b, c, d, a

In G_2 Euler circuit and Hamiltonian cycle are a, b, c, a.

Q.19 (a) ***Explain the Minimal Spanning Tree. Also write the Kruskal Algorithm for find Minimal Spanning tree.***

(b) ***Given the Graph in following figure. Apply Prim's algorithm to obtain the minimal spanning tree.***

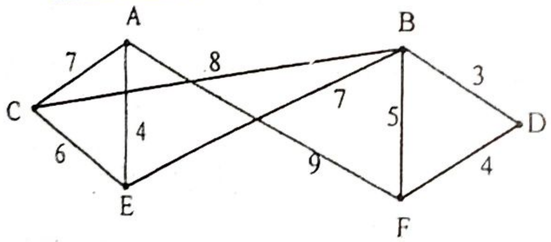


Fig. *[R.T.U. 2012]*

Ans. (a) Minimal Spanning Tree

If a connected weighted tree G, then its minimal spanning tree is a spanning tree of G such that the sum of the weights of its edges is minimum. For instance for the following graph of figure. The spanning tree, shown by thicker lines is the one of minimum weight.

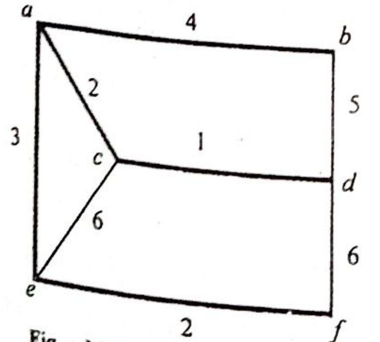


Fig. : Minimum Spanning Tree

Kruskal's algorithm

An algorithm in graph theory that finds a minimal spanning tree for a connected weighted graph. This means it finds a subset of the edges that forms a tree that includes every vertex, where the total weight of all the edges in the tree is minimized. If the graph is not connected, then it finds a minimal spanning forest (a minimal spanning tree for each connected component).

Kruskal's algorithm is an example of a greedy algorithm.

This algorithm was written by Joseph Kruskal in 1956.

An algorithm for computing a minimal spanning tree. It maintains a set of partial minimal spanning trees, and repeatedly adds the shortest edge in the graph whose vertices are in different partial minimal spanning trees.

Algorithm of finding minimal spanning tree by Kruskal's algorithm

Step 1 : Create a forest F (a set of trees), where each vertex in the graph is a separate tree

Step 2 : Create a set S containing all the edges in the graph

Step 3 : While S is nonempty

(a) remove an edge with minimum weight from S

(b) if that edge connects two different trees, then add it to the forest, combining two trees into a single tree

(c) otherwise discard that edge

At the termination of the algorithm, the forest has only one component and forms a minimal spanning tree of the graph.

For example determine the minimal spanning tree in the following graph by applying Kruskal's algorithm.

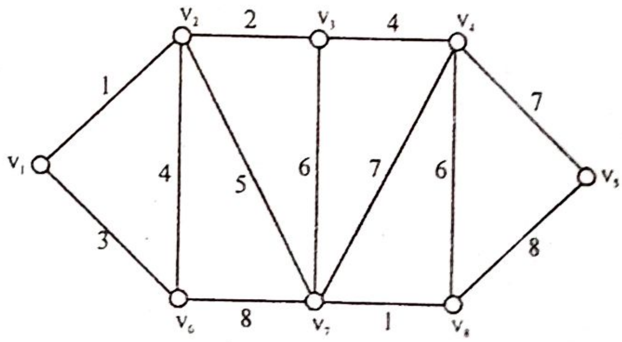
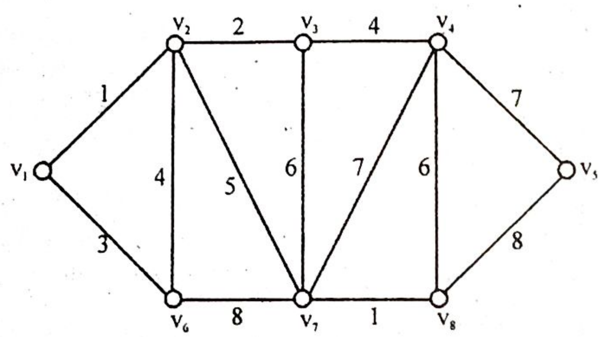


Fig.

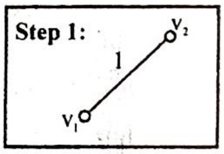
Using the above graph, here are the steps to the minimal spanning tree, using Kruskal's algorithm:

1. v_1 to v_2 –cost is 1 –add to tree
2. v_7 to v_8 –cost is 1 –add to tree
3. v_2 to v_3 – cost is 2 – add to tree

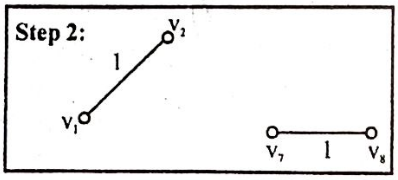
4. v_1 to v_6 – cost is 3 – add to tree
5. v_2 to v_6 – cost is 4 – reject because it forms a circuit
6. v_3 to v_4 – cost is 4 – add to tree
7. v_2 to v_7 – cost is 5 – add to tree
8. v_3 to v_7 – cost is 6 – reject because it forms a circuit
9. v_4 to v_8 – cost is 6 – reject because it forms a circuit
10. v_4 to v_7 – cost is 7 – reject because it forms a circuit
11. v_4 to v_5 – cost is 7 – add to tree
12. We stop here, because n – 1 edge has been added. We are left with the minimal spanning tree, with a total weight of 23.



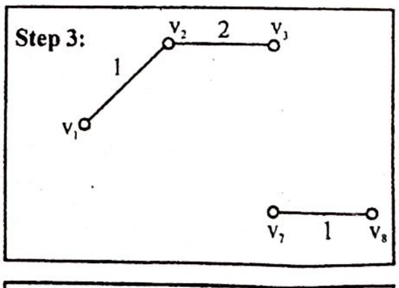
Step 1:



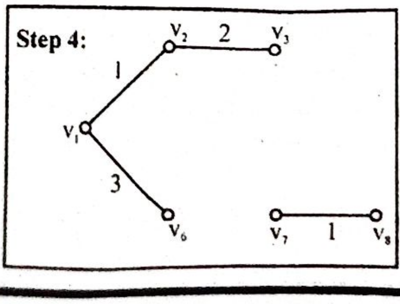
Step 2:



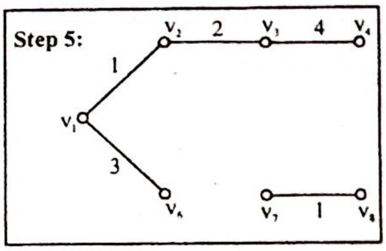
Step 3:



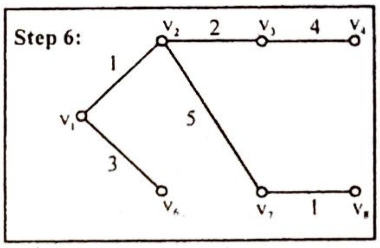
Step 4:



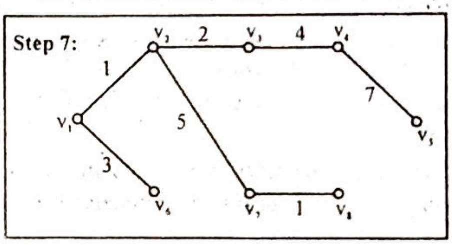
Step 5:



Step 6:



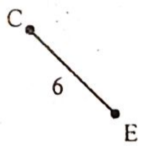
Step 7:



Ans. (b) (i) Start at C

CE is the lowest-weighted edge (6).

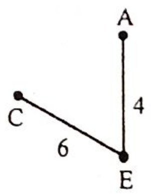
Draw it in.



(ii) From C or E

EA is the lowest-weighted edge (4).

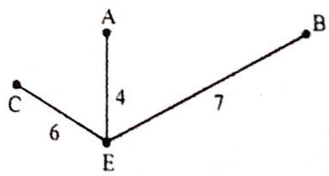
Draw it in.



(iii) From C, E or A

EB is the lowest-weighted edge (7).

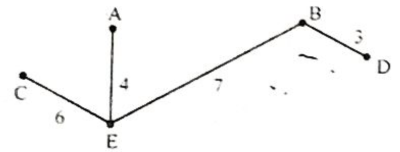
Draw it in.



(iv) From C, E, A or B

BD is the lowest-weighted edge (3).

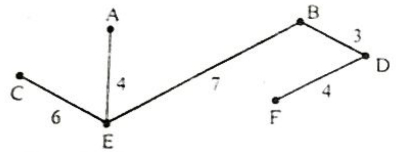
Draw it in.



(v) From C, E, A, B or D

DF is the lowest-weighted edge (4).

Draw it in.



- All vertices have now been joined.

The minimum spanning tree is determined.

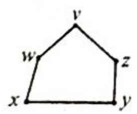
Minimum spanning tree length = 6 + 4 + 7 + 3 + 4

= 24 units.

Q.20 *Write a detailed note on Hamiltonian path and circuits with example.*

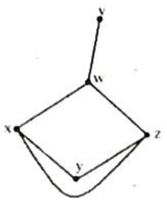
Ans. Hamiltonian Path and Circuits : A Hamiltonian circuit in a graph G is a circuit, that contains each vertex of G once (except for the starting and ending vertex, which occurs twice). A Hamiltonian path in G is a path (not a circuit) that contains each vertex of G once. Note that by deleting an edge in a Hamiltonian circuit we get a Hamiltonian path, so if a graph has a Hamiltonian circuit, then it also has a Hamiltonian path. The converse is not true, *i.e.*, *a* graph may have a Hamiltonian path but not a Hamiltonian circuit.

Example 1 : *Find a Hamiltonian circuit in the graph :*



Solution: *vwxyzv*

Example 2 : *Show that the following graph has a Hamiltonian path but no Hamiltonian circuit:*



Solution: *vwxyz* is a Hamiltonian path. There is no Hamiltonian circuit since no cycle goes through *v*.

In general it is not easy to determine if a given graph has a Hamiltonian path or circuit, although often it is possible to argue that a graph has no Hamiltonian circuit. For instance if G = (V, E) is a bipartite graph with vertex partition $\{V_1, V_2\}$ (so that each edge in G connects some vertex in V_1 to some vertex in V_2), then G cannot have a Hamiltonian circuit if $|V_1| \neq |V_2|$, because any path must contain alternatively vertices from V_1 and V_2, so any circuit in G must have the same number of vertices from each of both sets.

Edge Removal Argument : Another kind of argument consists of removing edges trying to make the degree of every vertex equal two. For instance in the graph of Fig. we cannot remove any edge because that would make the degree of *b, e* or *d* less than 2, so it is impossible to reduce the degree of *a* and *c*. Consequently that graph has no Hamiltonian circuit.

Dirac's Theorem : *If G is a simple graph with n vertices with $n \geq 3$ such that the degree of every vertex in G is at least n/2, then G has a Hamiltonian circuit.*

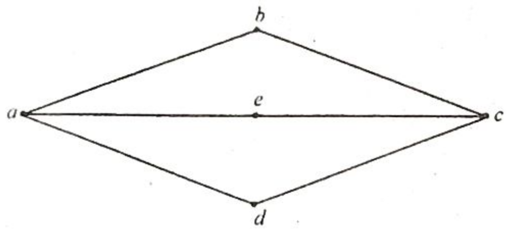
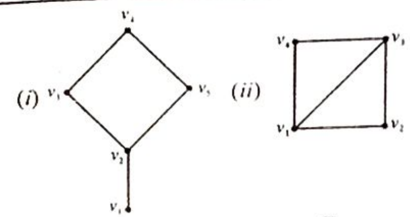


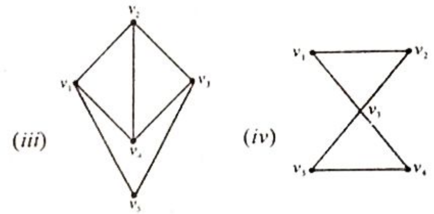
Fig. : Graph without Hamiltonian Circuit

Ore's Theorem : *If G is a simple graph with n vertices with $n \geq 3$ such that $deg(u) + deg(v) \geq n$ for every pair of non-adjacent vertices u and v in G, then G has a Hamiltonian circuit.*

Example 3 : *Which of the following graphs has a Hamiltonian path or cycle :*



G_1 G_2



G_3 G_4

Solution: (i) Graph G_1 has a Hamiltonian path v_1, v_2, v_3, v_4, v_5 but no Hamiltonian cycle.

(ii) Graph G_2 has a Hamiltonian cycle v_1, v_2, v_3, v_4, v_1.

(iii) Graph G_3 has a Hamiltonian cycle $v_1, v_5, v_3, v_4, v_2, v_1$.

(iv) Graph G_4 has no Hamiltonian path.

Q.21 *Explain shortest path problem with example.*

Ans. Shortest Path : Shortest path between two vertices in a graph is the path of minimum length. Thus, if :

(a) The graph is without weights, the length of path denotes the number of edges in the path and shortest path between two vertices is the path with least number of edges.

(b) The graph is weighted graph, the shortest path between two vertices is the path of minimum length (weight).

Shortest Path Problem : With each edge *e* of G let there associate a real number *w*(*e*), called its weight. Then G, together with these weights on its edges, is called a weighted graph.

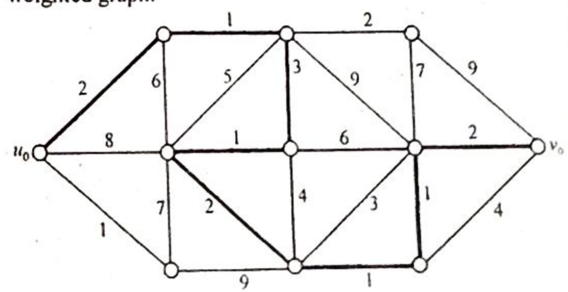


Fig. 1 : A (u_0, v_0) Path of Minimum Weight